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Fiscal Sustainability in a New Keynesian Model

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Abstract: There has been a wealth of recent work deriving optimal monetary policy utilising New Neo-Classical Synthesis (NNCS) models based on nominal inertia. Such models typically abstract from the impact of monetary policy on the government's finances, by assuming that consumers are infinitely-lived and taxes are lump-sum such that Ricardian Equivalence holds. In this paper, in the context of a sticky-price NNCS model, we assume that the government must adjust spending and/or distortionary taxation to satisfy its intertemporal budget constraint. We then consider optimal monetary and fiscal policies under discretion and commitment in the face of technology, preference and cost-push shocks. We find that the optimal precommitment policy implies a random walk in the steady-state level of debt, generalising earlier results that involved only a single fiscal instrument. In the case of negative fiscal shocks this implies permanently higher taxation and lower output and government spending to support the new steady-state debt stock, but the optimal combination of these variables will ensure a zero rate of inflation under commitment. We also find that the time-inconsistency in the optimal precommitment policy is such that governments are tempted, given inflationary expectations, to raise taxation to reduce the ultimate debt burden they need to service. Since taxation is a distortionary labour income tax, this aggressive raising of taxation raises firms' marginal costs and fuels inflation. We show that this temptation is only eliminated if following shocks, the new steady-state debt is equal to the original, first-best, debt level. This implies that under discretionary policy the random walk result is overturned: debt will always be returned to this initial steady-state even although there is no explicit debt target in the government's objective function. In a series of numerical simulations we show that the welfare consequences of introducing debt are negligible for precommitment policies, but can be significant for discretionary policy.

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1 Overview

There has been a wealth of recent work deriving optimal monetary policy utilising New Neo-Classical Synthesis (NNCS) models. Such models introduce a stabilisation role for monetary policy by assuming a nominal friction, often in the form of overlapping price contracts of the Cavlo (1983) type. This friction implies the optimal rate of inflation is typically close to zero to avoid exacerbating the welfare costs of relative price distortions[1]. However, such models usually only introduce fiscal policy as a convenient device through which to ensure the steady-state is efficient, ignoring the impact of monetary policy on the government's finances. Monetary policy can affect the government's budget constraint through various channels - through siegniorage, by affecting debt service costs, and by affecting the size of the tax base and need for fiscal transfers when prices are sticky. Most monetary policy models typically abstract from such effects, by assuming that consumers are infinitely-lived and taxes are lump-sum such that Ricardian Equivalence holds.

In this paper we relax this assumption and, in the context of a NNCS model, require that the government must adjust spending and/or distortionary taxation to satisfy its intertemporal budget constraint. We then consider optimal monetary and fiscal policies under discretion and commitment in the face of technology, preference and cost-push shocks and assess the extent to which ignoring the fiscal consequences of monetary policy affects the usual description of optimal monetary policy. Papers which do introduce debt and distortionary taxes to this model include Benigno and Woodford (2003) and Schmitt-Grohe and Uribe (2004), however, in both cases they consider commitment policies, whereas we also focus of the nature of the time-inconsistency problem and its implication for discretionary policy-making. The time-inconsistency problem in this context is particularly relevant given that policy makers are prepared to place quite tough constraints on their fiscal policies (see for example, the Stability and Growth Pact) and these constraints are often far from credible.

We find that the optimal precommitment policy implies a random walk in the steady-state level of debt, generalising earlier results that involved only a single fiscal instrument. However, our analysis of the time-inconsistency problem reveals that governments are, in the face of negative fiscal shocks and existing inflationary expectations, tempted to raise taxes to reduce the future burden of servicing the debt stock. These higher taxes will raise labour costs and fuel

[1] This is in contrast to the flexible price literature where optimal monetary policy is typically described by the Friedman rule which involves a steady deflation which underpins a nominal interest rate of zero.

inflation which will also serve to reduce the debt stock through surprise inflation. However, even if debt is real this temptation to raise taxes beyond their precommitment level, given economic agents expectations, remains. This implies that, under discretionary policy debt will always be returned to its initial steady-state to eliminate this temptation, and debt no longer follows a random walk. In adding debt to the New Keynesian model, the problem with discretionary policy (relative to commitment) is not that it fails to stabilise the debt stock, but that it is overzealous in doing so. In a series of numerical simulations we show that the welfare consequences of introducing debt are negligible for precommitment policies, but can be significant for discretionary policy.

There has been previous work examining time-consistency problems in the presence of debt, although this has been in models based on flexible prices - to the extent that the surprise inflation is not merely used to offset the fiscal effects of shocks then a potential time-inconsistency problem exists. However, this time-inconsistency problem need not imply a postive inflation bias, but can be consistent with the Friedman rule (Obstfeld, 1991,1997), or, for alternative preferences, a positive steady-state debt level where the time-inconsistency problem has been eliminated (Ellison and Rankin (2006)). Nevertheless, given that optimal monetary policy results under flexible prices, such as the Friedman rule, are not robust to the introduction of sticky prices, it is important to extend this analysis to the case of sticky prices and distortionary taxes.

The plan of the paper is as follows. In Section 2 we outline our model in which consumers supply labour to imperfectly competitive firms who are only able to change prices at random intervals of time. Workers' labour income is taxed. In Section 3 we derive a second-order approximation to welfare for these consumers. This is important since the effective rejection of the Friedman rule in sticky-price models relies on the dominance of the welfare costs of price-distortions relative to the costs reducing the inflation tax. We eliminate the usual inflationary bias caused by an inefficiently low level of steady-state output due to imperfect competition and distortionary taxes, by introducing a subsidy financed by lump-sum taxes. However, we do not allow further use of lump-sum taxes to finance government spending and ensure fiscal solvency following shocks - instead governments must adjust spending and/or income taxes to ensure fiscal sustainability. This allows us to focus on the time inconsistency caused by the need to stabilise debt. In Section 4, we describe the optimal precommitment policy and analyse the time-inconsistency inherent in that policy, before computing the discretionary policy in Section 5. This then informs the simulation results in section 6, which reveal that operating under discretion overturns the usual random walk result and can potentially generate significant welfare costs.

2 The Model

This section outlines our model. We examine the households' problem initially, before turning to the firms' problem.

2.1 Households

There are a continuum of households of size one, who differ in that they provide differentiated labour services to firms in their economy. However, we shall assume full asset markets, such that, through risk sharing, they will face the same budget constraint. As a result the typical household will seek to maximise the following objective function,

$$E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t, G_t; \xi_t; \xi_t^N) \tag{1}$$

where C,G and N are a consumption aggregate, a public goods aggregate, and labour supply respectively, and ξ is a time preference shock and ξ_t^N is a labour supply shock.

The consumption aggregate is defined as[2]

$$C = (\int_0^1 C(j)^{\frac{\epsilon_t - 1}{\epsilon_t}} dj)^{\frac{\epsilon_t}{\epsilon_t - 1}} \tag{2}$$

where j denotes the good's type or variety. The public goods aggregate takes the same form

$$G = (\int_0^1 G(j)^{\frac{\epsilon_t - 1}{\epsilon_t}} dj)^{\frac{\epsilon_t}{\epsilon_t - 1}} \tag{3}$$

The elasticity of substitution between varieties $\epsilon_t > 1$ is time varying as we wish to allow for cost-push/mark-up shocks.

The budget constraint at time t is given by

$$\int_0^1 P_t(j) C_t(j) dj + E_t\{Q_{t,t+1} D_{t+1}\} = \Pi_t + D_t + W_t N_t (1 - \tau_t) - T_t \tag{4}$$

where $P_t(j)$ is the price of variety j , D_{t+1} is the nominal payoff of the portfolio held at the end of period t, Π is the representative household's share of profits in the imperfectly competitive firms, W are wages, τ is an wage income tax rate, and T are lump sum taxes. $Q_{t,t+1}$ is the stochastic discount factor for one period ahead payoffs.

Households must first decide how to allocate a given level of expenditure across the various goods that are available. They do so by adjusting the share of a particular good in their consumption bundle to exploit any relative price differences - this minimises the costs of consumption. Optimisation of expenditure for any individual good implies the demand function given below,

$$C(j) = (\frac{P(j)}{P})^{-\epsilon_t} C \tag{5}$$

where we have price indices given by

$$P = (\int_0^1 P(j)^{1-\epsilon_t} dj)^{\frac{1}{1-\epsilon_t}} \tag{6}$$

[2] We drop the time subscript when all variables in an expression are dated in the same period and there is no possibility of confusion.

It follows that

$$\int_0^1 P(j)C(j)dj = PC \tag{7}$$

The budget constraint can therefore be rewritten as

$$P_tC_t + E_t\{Q_{t,t+1}D_{t+1}\} = D_t + W_tN_t(1-\tau_t) - T_t \tag{8}$$

2.1.1 Households' Intertemporal Consumption Problem

The first of the households intertemporal problems involves allocating consumption expenditure across time. For tractability assume that (1) takes the specific form

$$E_0\sum_{t=0}^{\infty}\beta^t(\frac{C_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma} + \chi\frac{G_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma} - \frac{{N_t}^{1+\varphi}\xi_t^{-\sigma}\xi_t^N}{1+\varphi}) \tag{9}$$

We can then maximise utility subject to the budget constraint (8) to obtain the optimal allocation of consumption across time,

$$\beta(\frac{C_t}{C_{t+1}})^{\sigma}(\frac{\xi_t}{\xi_{t+1}})^{\sigma}(\frac{P_t}{P_{t+1}}) = Q_{t,t+1} \tag{10}$$

Taking conditional expectations on both sides and rearranging gives

$$\beta R_tE_t\{(\frac{C_t}{C_{t+1}})^{\sigma}(\frac{\xi_t}{\xi_{t+1}})^{\sigma}(\frac{P_t}{P_{t+1}})\} = 1 \tag{11}$$

where $R_t = \frac{1}{E_t\{Q_{t,t+1}\}}$ is the gross return on a riskless one period bond paying off a unit of currency in $t+1$. This is the familiar consumption Euler equation which implies that consumers are attempting to smooth consumption over time such that the marginal utility of consumption is equal across periods (after allowing for tilting due to interest rates differing from the households' rate of time preference).

A log-linearised version of (11) can be written as

$$\widehat{C}_t + \widehat{\xi}_t = E_t\{\widehat{C}_{t+1} + \widehat{\xi}_{t+1}\} - \frac{1}{\sigma}(r_t - E_t\{\pi_{t+1}\}) \tag{12}$$

where hatted variables denote percentage deviations from steady-state, $r_t = R_t - \rho$ where $\rho = \frac{1}{\beta} - 1$, and $\pi_t = p_t - p_{t-1}$ is price inflation.

The second foc relates to their labour supply decision and is given by,

$$(1-\tau)\left(\frac{W}{P}\right) = N^{\varphi}C^{\sigma}\xi^N \tag{13}$$

Log-linearising implies,

$$\ln(1-\widehat{\tau}) + \widehat{w} = \varphi\widehat{N} + \sigma\widehat{C} + \widehat{\xi}^N \tag{14}$$

2.2 Allocation of Government Spending

The allocation of government spending across goods is determined by minimising total costs, $\int_0^1 P(j)G(j)dj$. Given the form of the basket of public goods this implies,

$$G(j) = (\frac{P(j)}{P})^{-\epsilon_t} G \tag{15}$$

2.3 Firms

The production function is linear, so for firm j

$$Y(j) = AN(j) \tag{16}$$

where $a = \ln(A)$ is time varying and stochastic. While the demand curve they face is given by,

$$Y(j) = (\frac{P(j)}{P})^{-\epsilon_t} Y \tag{17}$$

where $Y = \left[\int_0^1 Y(j)^{\frac{\epsilon_t - 1}{\epsilon_t}} dj\right]^{\frac{\epsilon_t}{\epsilon_t - 1}}$. The objective function of the firm is given by,

$$\sum_{s=0}^{\infty} (\theta_p)^s Q_{t,t+s} \left[\frac{P(j)_t}{P_{t+s}} Y(j)_{t+s} - \frac{W_{t+s}}{P_{t+s}} \frac{Y(j)_{t+s}(1-\varkappa)}{A} \right] \tag{18}$$

where $\varkappa$ is an employment subsidy which can be used to eliminate the steady-state distortion associated with monopolistic competition and distortionary income taxes (assuming there is a lump-sum tax available to finance such a subsidy). Using the demand curve for the firm's product,

$$\sum_{s=0}^{\infty} (\theta_p)^s Q_{t,t+s} \begin{bmatrix} \frac{P(j)_t}{P_{t+s}} (\frac{P(j)_t}{P_{t+s}})^{-\epsilon_t} Y_{t+s} \\ -\frac{W_{t+s}}{P_{t+s}} (\frac{P(j)_t}{P_{t+s}})^{-\epsilon_t} \frac{Y_{t+s}(1-\varkappa)}{A_{t+s}} \end{bmatrix} \tag{19}$$

The solution to this problem is given by,

$$\sum_{s=0}^{\infty} (\theta_p)^s Q_{t,t+s} \begin{bmatrix} (1-\epsilon_t) P_{t+s}^{-1} (\frac{P(j)_t}{P_{t+s}})^{-\epsilon_t} Y_{t+s} \\ +\epsilon_t \frac{W_{t+s}}{P_{t+s}} P(j)_t^{-\epsilon_t - 1} P_{t+s}^{\epsilon_t} \frac{Y_{t+s}(1-\varkappa)}{A_{t+s}} \end{bmatrix} \tag{20}$$

Solving for the optimal reset price, which is common across all firms able to reset prices in period t,

$$P_t^* = \frac{\sum_{s=0}^{\infty} (\theta_p)^s Q_{t,t+s} \left[\epsilon_t \frac{W_{t+s}}{P_{t+s}} P_{t+s}^{\epsilon_t} \frac{Y_{t+s}}{A_{t+s}} \right]}{\sum_{s=0}^{\infty} (\theta_p)^s Q_{t,t+s} \left[(\epsilon_t - 1) P_{t+s}^{-1} P_{t+s}^{\epsilon_t} Y_{t+s}(1-\varkappa) \right]} \tag{21}$$

While the price level evolves according to,

$$P_t = \left[(1-\theta_p) P_t^{*(1-\epsilon_t)} + \theta_p P_{t-1}^{1-\epsilon_t} \right]^{\frac{1}{1-\epsilon_t}} \tag{22}$$

Appendix 1 then details the derivation of the New Keynesian Phillips curve for price inflation which is given by,

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\theta_p\beta)(1-\theta_p)}{\theta_p}(\widehat{mc}_t + \widehat{\mu}_t) \tag{23}$$

where $\widehat{mc} = -a + \widehat{w} - \widehat{v}$ are the real log-linearised marginal costs of production, $\widehat{\mu}_t = \ln(\frac{\epsilon_t}{\epsilon_t - 1}) - \ln(\frac{\overline{\epsilon}}{\overline{\epsilon}-1})$is a mark-up shock representing the temporary deviation of the desired markup from its steady-state value, and $v = -\ln(1-\varkappa)$ is a transformation of the steady-state production subsidy. In the absence of sticky prices profit maximising behaviour implies, $mc_t = -\ln(\mu_t)$ where μ_t is the desired mark-up. Using the labour supply condition, $\ln(1-\widehat{\tau}) + \widehat{w} = \varphi\widehat{n} + \sigma\widehat{c} + \widehat{\xi}^N$, we can rewrite this as,

$$\pi_t = \beta E_t \pi_{t+1} + \gamma(-a_t + \varphi\widehat{n}_t + \sigma\widehat{c}_t + \widehat{\xi}_t^N - \widehat{v}_t - \ln(1-\widehat{\tau}_t) + \widehat{\mu}_t) \tag{24}$$

where $\gamma = \frac{(1-\theta_p\beta)(1-\theta_p)}{\theta_p}$. From the production funcion,

$$\pi_t = \beta E_t \pi_{t+1} + \gamma(-(1+\varphi)a_t + \varphi\widehat{y}_t + \sigma\widehat{c}_t + \widehat{\xi}_t^N - \widehat{v}_t - \ln(1-\widehat{\tau}_t) + \widehat{\mu}_t) \tag{25}$$

2.4 Equilibrium

Goods market clearing requires, for each good j,

$$Y(j) = C(j) + G(j) \tag{26}$$

which allows us to write,

$$Y(j) = (\frac{P(j)}{P})^{-\epsilon_t}[C+G] \tag{27}$$

Defining aggregate output as

$$Y = [\int_0^1 Y(j)^{\frac{\epsilon_t - 1}{\epsilon_t}} dj]^{\frac{\epsilon_t}{\epsilon_t - 1}} \tag{28}$$

allows us to write

$$Y = C + G \tag{29}$$

Log-linearising implies

$$\widehat{Y} = \theta\widehat{C} + (1-\theta)\widehat{G} \tag{30}$$

where we define $\theta = \frac{\overline{C}}{\overline{Y}}$.

2.5 Government Budget Constraint

Recall the representative consumer's budget constraint,

$$P_tC_t + E_t\{Q_{t,t+1}D_{t+1}\} \leq \Pi_t + D_t + W_tN_t(1-\tau_t) - T_t \tag{31}$$

D_{t+1} is a random variable, whose value depends on the state of the world in period $t+1$ i.e. it is the household's planned state-contingent wealth. There is a unique stochastic discount factor which has the property,

$$A_t = E_t[Q_{t,t+1}D_{t+1}] \tag{32}$$

where A_t is the end-of period nominal value of the household's portfolio of assets. If the household chooses to hold only risk-less one period bonds then this condition becomes,

$$D_{t+1} = R_tA_t$$

However, households will not only hold government bonds as they will wish to hold a complete set of contingent assets (given the stickiness in wage and price setting). The wealth D_{t+1}being transferred into the next period satisfies the bound,

$$D_{t+1} \geq -\sum_{T=t+1}^{\infty} E_{t+1}[Q_{t+1,T}(\Pi_T + W_TN_T(1-\tau_T) - T_T)] \tag{33}$$

with certainty, no matter what state of the world emerges. These series of borrowing constraints and flow budget constraints then defines the intertemporal budget constraint. It is normal to rule out no-Ponzi schemes which amount to,

$$\sum_{T=t}^{\infty} E_t[Q_{t,T}(\Pi_T + W(k)_TN(k)_T(1-\tau_T) - T_T)] < \infty \tag{34}$$

at each point in time across all possible states of the world. These can be combined to yield the intertemporal budget constraint (see Woodford, 2003, chapter 2, page 69),

$$\sum_{T=t}^{\infty} E_t[P_TC_T] \leq D_t + \sum_{T=t}^{\infty} E_t[Q_{t,T}(\Pi_T + W_TN_T(1-\tau_T) - T_T)] \tag{35}$$

Noting the equivalence between factor incomes and national output,

$$PY = WN + \Pi - \varkappa WN \tag{36}$$

we can rewrite the private sector's budget constraint as,

$$D_t = -\sum_{T=t}^{\infty} E_t[Q_{t,T}(P_TY_T - P_TC_T - W_TN_T(\tau_T - \varkappa) - T_T)] \tag{37}$$

Using the definition of aggregate demand,

$$D_t = -\sum_{T=t}^{\infty} E_t[Q_{t,T}(P_T G_T - W_T N_T(\tau_T - \varkappa) - T_i)] \tag{38}$$

In order to focus on the time-inconsistency problem associated with the introduction of debt and distortionary taxation to the NNCS model we introduce a steady-state subsidy (which offsets, in steady-state, the distortions caused by distortionary taxation and imperfect competition in wage and price setting). This subsidy is financed by lump-sum taxation and removes the usual desire on the part of policy makers to raise output above its natural level to compensate for these distortions. We shall then assume that lump-sum taxation cannot be used to alter this subsidy or to finance any other government activities, including the kind of spending and distortionary tax adjustments as stabilisation measures we are interested in. This implies that $W_T^i N_T^i \varkappa_i = T_T^i$ in all our economies at all points in time, allowing us to simplify the budget constraint to,

$$D_t = -\sum_{T=t}^{\infty} E_t[Q_{t,T}(P_{i,T} G_T^i - W_T^i N_T^i \tau_T^i)] \tag{39}$$

i.e. distortionary taxation and spending adjustments are required to service government debt as well as stabilise the economy. Rewriting in real terms and using government debt,

$$\frac{B_{t-1}}{P_{t-1}} \frac{P_{t-1}}{P_t} = \sum_{T=t}^{\infty} E_t[R_{t,T}(w_T^i N_T^i \tau_T^i - G_T^i)] \tag{40}$$

From the Euler equation we have,

$$\beta^{T-t}(\frac{C_t}{C_T})^\sigma(\frac{\xi_t}{\xi_T})^\sigma = Q_{t,T}\frac{P_T}{P_t} = R_{t,T} \tag{41}$$

allowing is to rewrite the budget constraint as,

$$\frac{B_{t-1}}{P_{t-1}} \frac{P_{t-1}}{P_t} = \sum_{T=t}^{\infty} E_t[\beta^{T-t}(\frac{C_t}{C_T})^\sigma(\frac{\xi_t}{\xi_T})^\sigma(w_T N_T \tau_T - G_T)] \tag{42}$$

In the steady-state,

$$\overline{b} = \frac{\overline{w}\overline{N}\overline{\tau} - \overline{G}}{1-\beta} \tag{43}$$

Log-linearising around this steady-state,

$$\widehat{b}_{t-1} - \pi_t = \beta E_t\{\widehat{b}_t - \pi_{t+1}\} + [\frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}(\widehat{w}_t + \widehat{N}_t + \widehat{\tau}_t) - \frac{\overline{G}}{\overline{b}}\widehat{G}_t] \tag{44}$$

3 Optimal policy

3.1 The Social Planner's Problem

In order to derive a welfare function for policy analysis we proceed in the following manner. Firstly, we consider the social planner's problem. We then contrast this with the outcome under flexible prices in order to determine the level of the steady-state subsidy required to ensure the model's steady-state is socially optimal. Finally, we construct a quadratic approximation to utility in our sticky-price/distortionary tax economy which assesses the extent to which endogenous variables differ from the flex-price equilibrium due to the nominal inertia and tax distortions present in the model.

The social planner is not constrained by the price mechanism and simply decides how to allocate consumption and production of goods within the economy. Since the social planner will produce equal quantities of all goods we can write the production technology as

$$Y = AN \tag{45}$$

and the resource constraint,

$$Y = C + G \tag{46}$$

The social planner maximises

$$\left(\frac{C_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma} + \chi\frac{G_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma} - \frac{{N_t}^{1+\varphi}\xi_t^{-\sigma}\xi_t^N}{1+\varphi}\right) \tag{47}$$

subject to these two constraints, which implies for Y,

$$(Y_t - G_t)^{-\sigma}\xi_t^{-\sigma} - Y_t^{\varphi}A_t^{-(1+\varphi)}\xi_t^N\xi_t^{-\sigma} = 0 \tag{48}$$

and for G,

$$-(Y_t - G_t)^{-\sigma}\xi_t^{-\sigma} + \chi {G_t}^{-\sigma}\xi_t^{-\sigma} = 0 \tag{49}$$

which implies,

$$(C_t)^{-\sigma} = \chi {G_t}^{-\sigma} \tag{50}$$

In the steady-state these reduce to,

$$(\overline{C}^*)^{-\sigma} = (\overline{Y}^*)^{\varphi} = (\overline{N}^*)^{\varphi} \tag{51}$$

and,

$$(\overline{C}^*)^{-\sigma} = \chi(\overline{G}^*)^{-\sigma} \tag{52}$$

where we introduce the '*' superscript to denote the efficient steady-state level of that variable. These can be log-linearised around this steady-state as,

$$-\sigma\widehat{C}_t^* - \sigma\widehat{\xi}_t = \varphi\widehat{Y}_t^* - (1+\varphi)a_t + \varphi\widehat{\xi}_t \tag{53}$$

and,

$$\widehat{C}_t^* = \widehat{G}_t^* \tag{54}$$

where we a mesauring the optimal deviation of the variable from the efficient steady-state in the face of shocks. From the national accounting identity the latter implies, $\widehat{C}_t^* = \widehat{G}_t^* = \widehat{Y}_t^*$.

3.2 Flexible Price Equilibrium

Profit-maximising behaviour implies that firms will operate at the point at which marginal costs equal marginal revenues,

$$
\begin{aligned}
-\ln(\mu_t) &= mc_t \qquad (55)\\
\left(1-\frac{1}{\epsilon_t}\right) &= \frac{(1-\varkappa_t)}{(1-\tau_t)}(N_t^n)^{(1+\varphi)}(C_t^n)^{\sigma}\xi_t^N \qquad (56)
\end{aligned}
$$

In the initial steady-state this reduces to,

$$\left(1-\frac{1}{\overline{\epsilon}}\right) = \frac{(1-\varkappa)}{(1-\overline{\tau})}(\overline{N}^n)^{\varphi}(\overline{C}^n)^{\sigma} \qquad (57)$$

If the subsidy $\varkappa$ is given by

$$(1-\varkappa) = (1-\frac{1}{\overline{\epsilon}})(1-\overline{\tau}) \qquad (58)$$

then

$$(\overline{C}^n)^{-\sigma} = (\overline{N}^n)^{\varphi} \qquad (59)$$

which is identical to the optimal level of employment in the efficient steady-state derived above. If the government implements the govt spending plans in line with the social planner's problem in steady-state then the flex price steady-state is the same as the efficient output level.

From the labour supply condition, if the subsidy is in place, then the steady-state real wage is given by,

$$\overline{w} = \frac{1}{1-\overline{\tau}} \qquad (60)$$

and government spending as a share of GDP is given by,

$$\frac{\overline{G}}{\overline{\overline{Y}}} = (1+\chi^{-\frac{1}{\sigma}})^{-1} \qquad (61)$$

The steady-state tax rate required to support a given debt to GDP ratio is given by,

$$\overline{\tau} = \frac{(1-\beta)\frac{\overline{B}}{\overline{Y}} + \frac{\overline{G}}{\overline{Y}}}{1+(1-\beta)\frac{\overline{B}}{\overline{Y}} + \frac{\overline{G}}{\overline{Y}}} \qquad (62)$$

The tax revenues relative to debt this implies are given by,

$$\frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}} = \frac{\frac{\overline{\tau}}{1-\overline{\tau}}}{\frac{\overline{B}}{\overline{Y}}} \qquad (63)$$

This is enough to define all log-linearised relationships dependent on model parameters and the initial debt to gdp ratio.

3.3 Social Welfare

Appendix 3 derives the quadratic approximation to utility

$$\Gamma = -\overline{N}^{1+\varphi}\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\{\sigma\theta(\widehat{C}_t-\widehat{C}_t^*)^2 \quad +\sigma(1-\theta)(\widehat{G}_t-\widehat{G}_t^*)^2+\varphi(\widehat{Y}_t-\widehat{Y}_t^*)^2+\frac{\epsilon_t}{\gamma}\pi_t^2\} + tip + O[3] \tag{64}$$

It contains quadratic terms in price inflation reflecting the costs of price dispersion induced by inflation in the presence of nominal inertia, as well as terms in the consumption, government spending and output gaps i.e. the difference between the actual value of the variable and its optimal value under flexible prices. The weights attached to each element are a function of deep model parameters. The key to obtaining this quadratic specification is in adopting an employment subsidy which eliminates the steady-state distortions caused by imperfect competition in labour and product markets as well as the steady-state impact of a distortionary income tax. It is important to stress that this subsidy only applies in the steady-state such that it cannot be used as a policy instrument to either stabilise the economy or the government's finances in the face of shocks.

3.4 Gap variables

We have derived welfare based on various gaps, so we now proceed to rewrite our model in terms of the same gap variables to facilitate derivation of optimal policy. The consumption Euler equation can be written in gap form as,

$$(\widehat{C}_t-\widehat{C}_t^*) = E_t\{(\widehat{C}_{t+1}-\widehat{C}_{t+1}^*)\} - \frac{1}{\sigma}((r_t-r_t^*)-E_t\{\pi_{t+1}\}) \tag{65}$$

where $r_t^* = \sigma\frac{1+\varphi}{\sigma+\varphi}(E_t\{a_{t+1}\}-a_t)+\sigma(E_t\{\widehat{\xi}_{t+1}\}-\widehat{\xi}_t)-\frac{\sigma}{\sigma+\varphi}(E_t\{\widehat{\xi}_{t+1}^N\}-\widehat{\xi}_t^N)$ is the natural rate of interest. (This comes from the fact that $\widehat{C}_t^* = \widehat{Y}_t^*$ and the definition of the efficient level of output).

While the NKPC can be written as,

$$\pi_t = \beta E_t\pi_{t+1} + \gamma(-(1+\varphi)a_t+\varphi\widehat{Y}_t+\sigma\widehat{C}_t+\widehat{\xi}_t^N+\frac{\overline{\tau}}{1-\overline{\tau}}\widehat{\tau}_t+\mu_t) \tag{66}$$

Using the definition of efficient output this can be written,

$$\pi_t = \beta E_t\pi_{t+1} + \gamma(\varphi(\widehat{Y}_t-\widehat{Y}_t^*)+\sigma(\widehat{C}_t-\widehat{C}_t^*)+\frac{\overline{\tau}}{1-\overline{\tau}}\widehat{\tau}_t+\mu_t) \tag{67}$$

and, followimg Benigno and Woodford (2003) we can define, $\frac{\overline{\tau}}{1-\overline{\tau}}\ \widehat{\tau}_t^* = \mu_t$. In other words we are defining our 'efficient' tax rate as the tax rate required to perfectly offset the impact of a cost-push shock[3]. If we had access to a lump-sum tax to finance the budget deficit then this would be the optimal tax rate.

[3] It should be noted that we could define the tax 'gap' as being the actual tax rate relative to any benchmark tax rate we choose, such as, for example, the initial steady-state tax rate. However, it is convenient to define the gap relative to the tax rate which offsets the impact of a cost-push shock on inflation.

However, given the need to finance the government liabilities through distortionary taxation, actual tax rates are likely to deviate from the level required to prefectly offset shocks and the Phillips curve can be rewritten as,

$$\pi_t = \beta E_t \pi_{t+1} + \gamma(\varphi(\widehat{Y}_t - \widehat{Y}_t^*) + \sigma(\widehat{C}_t - \widehat{C}_t^*) + \frac{\overline{\tau}}{1-\overline{\tau}}(\widehat{\tau}_t - \widehat{\tau}_t^*)) \tag{68}$$

Appendix 4 rewrites the budget constraint constraint in gap form as,

$$\widehat{b}_{t-1} - \pi_t = \beta E_t\{\widehat{b}_t - \pi_{t+1}\} + ps_t \tag{69}$$

with the primary surplus defined as,

$$ps_t = \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}[(1+\varphi)(\widehat{Y}_t - \widehat{Y}_t^*) + \frac{1}{1-\overline{\tau}}(\widehat{\tau}_t - \widehat{\tau}_t^*) + \sigma(\widehat{C}_t - \widehat{C}_t^*)] - \frac{\overline{G}}{\overline{b}}(\widehat{G}_t - \widehat{G}_t^*) - f_t \tag{70}$$

and $f_t = -(1-\beta)\frac{(1+\varphi)}{\sigma+\varphi}a_t + \frac{(1-\beta)}{\sigma+\varphi}\widehat{\xi}_t^N - \frac{\overline{w}\overline{N}}{\overline{b}}\mu_t$ capturing the extent to which the various shocks hitting our model have fiscal consequences.

4 Precommitment Policy

In this section we shall consider the precommitment policies for our model. The Lagrangian associated with the policy problem under commitment in the presence of a government budget constraint is given by,

$$\begin{aligned} L_t \quad = \quad & E_t \sum_{s=0}^{\infty} \beta^s [\sigma\theta(c_{t+s}^g)^2 + \sigma(1-\theta)(g_{t+s}^g)^2 + \varphi(y_{t+s}^g)^2 + \frac{\epsilon}{\gamma}\pi_{t+s}^2 \\ & +\lambda_{t+s}^{\pi}(\pi_{t+s} - \beta\pi_{t+s+1} - \gamma(\varphi y_{t+s}^g + \sigma c_{t+s}^g + \frac{\overline{\tau}}{1-\overline{\tau}}\tau_{t+s}^g) \\ & +\lambda_{t+s}^{y}(y_{t+s}^g - (1-\theta)g_{t+s}^g - \theta c_{t+s}^g) \\ & +\lambda_{t+s}^{b}(\widehat{b}_{t+s-1} - \pi_{t+s} - \beta(\widehat{b}_{t+s} - \pi_{t+s+1}) + \frac{\overline{G}}{\overline{b}} g_{t+s}^g \\ & -\frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}[(1+\varphi)(y_{t+s}^g) + \frac{1}{1-\overline{\tau}}(\widehat{\tau}_{t+s}^g) + \sigma c_{t+s}^g] + f_{t+s})] \end{aligned}$$

where λ_{t+s}^{π}, λ_{t+s}^{y}, and λ_{t+s}^{b} are the lagrange multipliers associated with the NKPC, the resource constraint and the government's budget constraint respectively. To simplify notation we have rewritten the gap variables in the form, $x_t^g = \widehat{X}_t - \widehat{X}_t^*$.

We shall initially consider the first-order conditions for periods $s > 0$ which apply in the same form whether or not we consider optimal or timeless commitment. The differences between these two forms of commitment which hinge on policy behaviour in the initial period $s = 0$, and will be considered in detail below. Accordingly, the first-order conditions from this optimisation are given by the following set of equations. Firstly for consumption,

$$2\sigma\theta c_{t+s}^g - \gamma\sigma\lambda_{t+s}^{\pi} - \theta\lambda_{t+s}^{y} - \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}\sigma\lambda_{t+s}^{b} = 0 \tag{71}$$

government spending,

$$2\sigma(1-\theta)g^g_{t+s} - (1-\theta)\lambda^y_{t+s} + \frac{\overline{G}}{\overline{b}}\lambda^b_{t+s} = 0 \tag{72}$$

the output gap,

$$2\varphi y^g_{t+s} - \gamma\varphi\lambda^\pi_{t+s} + \lambda^y_{t+s} - \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}(1+\varphi)\lambda^b_{t+s} = 0 \tag{73}$$

debt,

$$E_t\lambda^b_{t+s} - \lambda^b_t = 0 \tag{74}$$

inflation,

$$2\frac{\epsilon}{\gamma}\pi_{t+s} + \Delta\lambda^\pi_{t+s} - \Delta\lambda^b_{t+s} = 0 \tag{75}$$

and taxation,

$$-\frac{\overline{\tau}}{1-\overline{\tau}}\gamma\lambda^\pi_{t+s} - \frac{\overline{w}\overline{N}}{\overline{b}}\frac{\overline{\tau}}{1-\overline{\tau}}\lambda^b_{t+s} = 0 \tag{76}$$

The foc for debt implies that the lagrange-multiplier for debt follows a random walk and this will underpin the random walk for debt result derived below. Combining this foc with the foc for inflation and the tax rate implies that, in the absence of new information, inflation is zero. From the NKPC, this in turn implies the following income tax rule,

$$\varphi y^g_{t+s} + \sigma c^g_{t+s} + \frac{\overline{\tau}}{1-\overline{\tau}}\tau^g_{t+s} = 0 \tag{77}$$

Solving the remaining focs implies the following relationships between gapped variables and the lagrange multiplier associated with debt,

$$y^g_{t+s} = -\frac{(\varphi+\sigma) - (1-\beta)\frac{\overline{B}}{\overline{Y}}}{2(\varphi+\sigma)\frac{\overline{B}}{\overline{Y}}}\lambda^b_t = -a_1\lambda^b_t \tag{78}$$

for consumption,

$$c^g_{t+s} = -\frac{(\varphi+\sigma)(\frac{\sigma-\theta(1-\theta)}{\theta\sigma}) - (1-\beta)\frac{\overline{B}}{\overline{Y}}}{2(\varphi+\sigma)\frac{\overline{B}}{\overline{Y}}}\lambda^b_t = -a_2\lambda^b_t \tag{79}$$

and government spending,

$$g^g_{t+s} = -\frac{(\varphi+\sigma)\frac{\theta}{\sigma} - (1-\beta)\frac{\overline{B}}{\overline{Y}}}{2(\varphi+\sigma)\frac{\overline{B}}{\overline{Y}}}\lambda^b_t = -a_3\lambda^b_t \tag{80}$$

The constancy of these various real gaps implies that monetary policy is set such that interest rates are consistent with the natural rate of interest. It is clear from these definitions that the coefficients, $a_i, i = 1, 2, 3$ are positive provided the initial steady-state debt stock satisfies the following conditions,

$$(1-\beta)\frac{\overline{B}}{\overline{\overline{Y}}} < (\sigma+\varphi)\frac{\theta}{\sigma} \tag{81}$$

$$(1-\beta)\frac{\overline{B}}{\overline{\overline{Y}}} < (\varphi+\sigma)(\frac{\sigma-\theta(1-\theta)}{\theta\sigma}) \tag{82}$$

$$(1-\beta)\frac{\overline{B}}{\overline{\overline{Y}}} < (\sigma+\varphi) \tag{83}$$

For plausible steady-state debt/GDP ratios all variants of this condition will hold[4], implying that that when $\lambda_t^b > 0$, y_{t+s}^g, c_{t+s}^g and g_{t+s}^g will all be negative, which implies that $\tau_{t+s}^g > 0$.The converse is true when $\lambda_t^b < 0$.

It is helpful to rewrite the the definition of the primary surplus, (70), in terms of the value of the lagrange multiplier associated with the government's budget constraint by substituting for the tax rule which applies after the initial period,

$$ps_{t+s} = \frac{\overline{y}}{\overline{b}}[(\frac{\overline{\tau}}{1-\overline{\tau}} - \varphi)y_{t+s}^g - \sigma c_{t+s}^g - (1-\theta)g_{t+s}^g] - f_{t+s} \tag{84}$$

Using the expressions relating the gap variables to the lagrange-multiplier this can be re-written as,

$$ps_{t+s} = \Psi\lambda_t^b - f_{t+s} \tag{85}$$

where $\Psi = (\frac{\overline{B}}{\overline{\overline{Y}}})^{-1}[(\varphi - \frac{\overline{\tau}}{1-\overline{\tau}})a_1 + \sigma a_2 + (1-\theta)a_3] > 0$, again for debt not too large[5].

4.1 Timelessly Optimal Policy

Under the timelessly optimal policy we can impose the same policy in the initial period, so that, from (85) and noting that inflation is zero under this policy, the government budget constraint will evolve according to,

$$\widehat{b}_{t-1} = E_t\sum_{s=0}^{\infty}\beta^s(-f_{t+s} + \Psi\lambda_t^b) \tag{86}$$

$$= (\frac{\Psi}{1-\beta})\lambda_t^b - E_t\sum_{s=0}^{\infty}\beta^s f_{t+s} \tag{87}$$

Solving for the lagrange multiplier,

$$(\frac{\Psi}{1-\beta})\lambda_t^b = \widehat{b}_{t-1} + E_t\sum_{s=0}^{\infty}\beta^s f_{t+s} \tag{88}$$

[4] For the parmater values adopted in the simulation section below, the annualised steady-state debt to GDP ratio would have to exceed 2812.5% for this condition to be violated.

[5] For the parameter values adopted in the simulation section below it is not possible for this coefficient to be negative for any positive debt to gdp ratio.

In other words the lagrange multiplier is proportional to the initial debt disequilibrium and the discounted value of any fiscal deterioriation implied by the various shocks expected to hit the economy. The new steady-state government surplus will be given by (85) implying, in the absence of further unexpected shocks, a new steady-state debt-gdp ratio of,

$$\bar{b}^{TC} = (\frac{\Psi}{1-\beta})\lambda_t^b = \widehat{b}_{t-1} + E_t \sum_{s=0}^{\infty} \beta^s f_{t+s} \tag{89}$$

i.e. there is a random walk in the steady-state of debt under the timelessly optimal precommitment policy, such that the debt stock fully incorporates the fiscal consequences of shocks. The permanent changes in taxation and government spending that occur under the timelessly optimal policy do not attempt to undo the fiscal consequences of shocks, but merely ensure that this new steady-state debt to gdp ratio is sustainable.

To summarise, the consequences of shocks for the welfare-relevant gap variables under the timelessly optimal policy depend solely on their fiscal impact. Monetary policy ensures that nominal interest rates are consistent with the natural rate of interest. Fiscal policy then offsets any cost-push shocks, but can only do so imperfectly because of the need to satisfy the intertemporal budget constraint. Both policies jointly ensure that inflation is always zero, but fiscal instruments have to be adjusted to ensure solvency. The optimal policy is to allow the fiscal effects of shocks to be fully reflected in the debt stock and to only adjust fiscal instruments (government spending gaps and the tax gap) to the extent required to support the new steady-state debt stock.

4.2 Commitment Policy and Time-Inconsistency

In this section we consider the case where the policy maker exploits the fact that expectations are given in the initial period. By contrasting the solution in the initial period to that which follows we can highlight the nature of the time-inconsistency problem facing policy makers, which will help generate intuition for the outcome under discretion. In the initial period, $s = 0$, assuming we are not imposing the timelessly optimal policy, the initial values of the lagrange multipliers associated with the problem will be zero. Since the only foc which is dynamic in the lagrange multipliers is that for inflation, in the initial period inflation will be determined by,

$$2\frac{\epsilon}{\gamma}\pi_{t+s} + \lambda_{t+s}^{\pi} - \widetilde{\lambda}_{t+s}^{b,j} = 0 \tag{90}$$

where $\widetilde{\lambda}_t^{b,j}$ is the lagrange-multiplier associated with the government's budget constraint under optimal (non-timeless) commitment where j = real,nom depending on whether debt is real or nominal. From the foc for taxation this can be rewritten as,

$$2\epsilon\pi_t = (\frac{\overline{w}\overline{N}}{\overline{b}} + \gamma)\widetilde{\lambda}_t^{b,j} \tag{91}$$

This captures the extent to which fiscal stress generates inflation in the initial period. (Note that if debt was real rather than nominal there would still be a time inconsistency problem implying inflation in the initial period is still given by this expression, although the size of the lagrangian associated with the budget constraint will be different - see below.)

The tax rule in the initial period is given by,

$$\gamma(\varphi y_t^g + \sigma c_t^g + \frac{\overline{\tau}}{1-\overline{\tau}}\tau_t^g) = \frac{1}{2\epsilon}(\frac{\overline{w}\overline{N}}{\overline{b}} + \gamma)\widetilde{\lambda}_t^{b.j} \tag{92}$$

implying that the initial period's inflation rate is given by,$(\frac{\overline{w}\overline{N}}{\overline{b}}+\gamma)\widetilde{\lambda}_t^{b,j}$. In other words, in the initial period, given inflationary expectations, the fiscal authorities are tempted to move the tax instrument by more than is necessary to simply support a steady-state debt stock that fully reflects the fiscal consequences of the shocks. This moves the steady-state debt stock closer to its optimal value. By doing so the fiscal authorities reduce the costs of servicing the ultimate steady-state debt stock, but the initial changes in the tax instrument also generates additional unexpected inflation. Given this behaviour in the initial period, the initial government surplus is given by,

$$ps_t = \Psi_0 \widetilde{\lambda}_t^{b.j} - f_t \tag{93}$$

where $\Psi_0 = \Psi + \frac{(1+(1-\theta)+(1-\beta)\frac{\overline{b}}{\overline{y}})(1+(1-\theta)+(1-\beta)\frac{\overline{b}}{\overline{y}}+\gamma\frac{\overline{b}}{\overline{y}})}{2\gamma\epsilon_t(\frac{\overline{b}}{\overline{y}})^2} > \Psi$. Therefore, when $\widetilde{\lambda}_t^{b,j} > 0$ taxes rise in the initial period, fueling inflation, taxes then fall back to a level which is still above their initial steady-state value while output, consumption and government spending have all fallen relative to their original steady-state values. Note that this inflation is a fiscal phenomenon in the sense that higher taxes raise marginal costs, causing inflation, while monetary policy will ensure that interest rates are in line with the natural rate of interest and are, in that sense, neither inflationary nor deflationary.

In order to determine the size of the lagrange multiplier assocated with the government's IBC, we need to substitute these expressions in the intertemporal budget constraint. This calculation varies according to whether or not debt is real or nominal, since in the later case inflation in the initial period can deflate the real value of the debt. Accordingly in the case of real debt the lagrange multiplier is defined by,

$$\widehat{b}_{t-1} - E_{t-1}\pi_t = (\Psi_0 + \frac{\Psi}{\beta^{-1}-1})\widetilde{\lambda}_t^{b,\text{real}} - E_t\sum_{s=0}^{\infty}\beta^s f_{t+s} \tag{94}$$

and surprise inflation will not deflate the real value of the debt stock, while in the case of nominal debt we need to take account of the impact of the initial period's inflation on the debt stock,

$$\widehat{b}_{t-1} - \pi_t = (\Psi_0 + \frac{\Psi}{\beta^{-1}-1})\widetilde{\lambda}_t^{b,nom} - E_t\sum_{s=0}^{\infty}\beta^s f_{t+s} \tag{95}$$

Using the expression for the initial rate of inflation and solving for the lagrange-multiplier for the nominal debt case yields,

$$(\frac{\overline{w}\overline{N}}{\bar{b}} + \gamma + \Psi_0 + \frac{\Psi}{\beta^{-1} - 1})\widetilde{\lambda}_t^{b,nom} = \widehat{b}_{t-1} + E_t \sum_{s=0}^{\infty} \beta^s f_{t+s} \tag{96}$$

In other words the lagrange-multiplier is again proportional to the sum of the initial debt-disequilibrium and the expected discounted value of the fiscal effects of shocks. However, cet. par. the value of the multiplier will not be as large when the policy maker exploits fixed expectations in the initial period to raise additional tax revenue and deflate the debt. This implies that, in the case of a shock with negative fiscal consequences, output, consumption and government spending will not fall by as much, and taxes will not need to rise by as much to support the new steady-state debt stock, which is lower than it would be under timelessly optimal policy. The new steady-state debt stock under (non-timeless) commitment for the case of nominal and real debt are given by,

$$\overline{b}^{C,nom} = (\frac{\Psi}{1-\beta})\widetilde{\lambda}_t^{b,nom} < \overline{b}^{C,\text{real}} = (\frac{\Psi}{1-\beta})\widetilde{\lambda}_t^{b,\text{real}} < \widehat{b}_{t-1} + E_t \sum_{s=0}^{\infty} \beta^s f_{t+s} \tag{97}$$

and since $\widetilde{\lambda}_t^{b,j}$ is lower than under the timeless perspective (λ_t^b), some of the fiscal consequences of the shocks will be undone by the policy implemented in the first period. This will be greater in the case of nominal debt than real debt since the aggressive fiscal policy is even more effective when debt deflation is possible. It is also interesting to note from (91) that inflation in the initial period is lower when debt is nominal as the combined effects of debt deflation and higher taxes moderate the need to raise taxes (and fuel inflation) relative to the case of real debt.

We are now in a position to describe the response to shocks under both forms of commitment. Shocks only have an effect on welfare-relevant gap variables to the extent that they have fiscal repercussions, the financing of which limits the extent to which monetary and fiscal policy can achieve the first-best solution. Under timeless commitment inflation is always zero. Policy allows the fiscal effects of shocks to be fully reflected in the debt stock and to only adjust fiscal instruments (government spending gaps and the tax gap) to the extent required to support the new steady-state debt stock. Under timeless commitment it does not matter whether debt is denominate in real or nominal terms. Time inconsistent (non-timeless) optimal commitment policy improves welfare further by exploiting the Phillips curve to change taxes and (through higher marginal costs) inflation in the initial period relative to the timeless case, but thereafter it also adjusts fiscal instruments only to the extent required to support the new steady-state debt stock. The generating of inflation in the inirtial period also means that it matters whether debt is real or nominal. The initial period change in taxes means that the increase in steady state debt (and associated changes in other variables) will be less under non-timeless commitment, and will be lower still when debt is nominal. In both cases, if shocks raise debt, then steady state taxes are higher, and steady state government spending, private consumption and output are all lower. Debt will slowly evolve until it reaches a new steady-state value consistent with the higher taxes, lower government spending and reduced consumption and output. The initial increase in taxes and inflation under the time inconsistent policy drives the outcome under discretion.

5 Discretionary Policy

In attempting to obtain the discretionary policy solution we must contend with the existence of forward and backward-looking constraints. Our model is of the general linear-quadratic form, by construction. Therefore we can write the evolution of inflation as follows,

$$E_t \boldsymbol{\pi}_{t+1} = \mathbf{A0}\boldsymbol{\pi}_t + \mathbf{A1}\mathbf{S}_{t-1} + \mathbf{A2}\mathbf{u}_t \tag{98}$$

where

$$\begin{aligned} \mathbf{A0} &= \left[\frac{1}{\beta}\right], \mathbf{A1} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and,} \\ \mathbf{A2} &= \begin{bmatrix} -\frac{\gamma(\varphi\theta+\sigma)}{\beta\theta} & -\frac{\gamma\overline{\tau}}{\beta(1-\overline{\tau})} & \frac{\gamma\sigma(1-\theta)}{\beta\theta} \end{bmatrix} \end{aligned}$$

and $\mathbf{S}_{t-1} = \begin{bmatrix} \widehat{b}_{t-1} \\ a_{t-1} \\ \widehat{\xi}_t^N \\ \mu_t \end{bmatrix}$ and $\mathbf{u}_t = \begin{bmatrix} y_t^g \\ \tau_t^g \\ g_t^g \end{bmatrix}$ are the vectors of state and control variables respectively.

There is a similar formula describing the evolution of the state variables[6],

$$\mathbf{S}_t = \mathbf{B0}\boldsymbol{\pi}_t + \mathbf{B1}\mathbf{S}_{t-1} + \mathbf{B2}\mathbf{u}_t \tag{99}$$

where

$$\begin{aligned} \mathbf{B0} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{B1} = \begin{bmatrix} \frac{1}{\beta} & -(1-\beta)\frac{(1+\varphi)}{\sigma+\varphi}\rho_a & \frac{(1-\beta)}{\sigma+\varphi}\rho_N & -((2-\theta)+(1-\beta)(\frac{\overline{b}}{\overline{y}})^{-1})\rho_\mu \\ 0 & \rho_a & 0 & 0 \\ 0 & 0 & \rho_N & 0 \\ 0 & 0 & 0 & \rho_\mu \end{bmatrix}, \text{ and,} \\ \mathbf{B2} &= \begin{bmatrix} -\frac{\gamma(\varphi\theta+\sigma)}{\beta\theta} - \frac{\overline{\tau}(\theta+\varphi\theta+\sigma)}{(1-\overline{\tau})\frac{\overline{b}}{\overline{y}}\theta} & -\frac{\gamma\overline{\tau}}{\beta(1-\overline{\tau})} - \frac{\overline{\tau}}{\beta(1-\overline{\tau})^2\frac{\overline{b}}{\overline{y}}} & \frac{\gamma\sigma(1-\theta)}{\beta\theta} + \frac{(1-\theta)(\theta(1-\overline{\tau})+\overline{\tau}\sigma}{\beta(1-\overline{\tau})\frac{\overline{b}}{\overline{y}}\theta} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

In this simulation section below we consider the impact of different shocks with different degrees of persistence. The first problem we face is in formulating

[6] In this section we make the empirically plausible assumption that debt is denominated in nominal terms.

a recursive problem when our model contains expectations of future variables. However, since we have a linear-quadratic formula we can hypothesize a solution of the form,

$$\boldsymbol{\pi}_t = \mathbf{F}\mathbf{S}_{t-1} \tag{100}$$

where $\mathbf{F} = \begin{bmatrix} f1 & f2 & f3 & f4 \end{bmatrix}$ is a $1x4$ vector of undefined constants. Leading this forward one period and utilising the equation describing the evolution of the state variables, we can write,

$$\mathbf{FB0}\boldsymbol{\pi}_t + \mathbf{FB1b}_{t-1} + \mathbf{FB2u}_t = \mathbf{A0}\boldsymbol{\pi}_t + \mathbf{A1b}_{t-1} + \mathbf{A2u}_t \tag{101}$$

Solving for inflation,

$$\boldsymbol{\pi}_t = \mathbf{C1b}_{t-1} + \mathbf{C2u}_t \tag{102}$$

where

$$\mathbf{C1} \equiv [\mathbf{FB0} - \mathbf{A0}]^{-1}[\mathbf{A1}{-}\mathbf{FB1}]$$

and,

$$\mathbf{C2} \equiv [\mathbf{FB0} - \mathbf{A0}]^{-1}[\mathbf{A2} - \mathbf{FB2}]$$

We can similarly eliminate endogenous variables from the evolution of the state variables,

$$\mathbf{S}_t = \mathbf{D1S}_{t-1} + \mathbf{D2u}_t \tag{103}$$

where

$$\mathbf{D1} \equiv [\mathbf{B0C1} + \mathbf{B1}]$$

and,

$$\mathbf{D2} \equiv [\mathbf{B0C2} + \mathbf{B2}]$$

The Bellman equation for this problem can then be written as,

$$V(\mathbf{S}_{t-1}) = \underset{\mathbf{u}_t}{Min}(\boldsymbol{\pi}_t\mathbf{R}\boldsymbol{\pi}_t + \mathbf{u}_t'\mathbf{Q}\mathbf{u}_t) + \beta E_t V(\mathbf{S}_t) \tag{104}$$

subject to,

$$\boldsymbol{\pi}_t = \mathbf{C1S}_{t-1} + \mathbf{C2u}_t \tag{105}$$

and,

$$\mathbf{S}_t = \mathbf{D1S}_{t-1} + \mathbf{D2u}_t \tag{106}$$

where $\mathbf{R} = \left[\frac{\varepsilon}{\gamma}\right]$ and $\mathbf{Q} = \begin{bmatrix} \frac{\varphi\theta+\sigma}{\theta} & 0 & -2\frac{(1-\theta)\sigma}{\theta} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{(1-\theta)\sigma}{\theta} \end{bmatrix}$ so that we now have a fully recursive formulation of the problem.

The first-order conditions with respect to the control variables from this optimisation are then given by,

$$2\mathbf{C2}'\mathbf{R}\boldsymbol{\pi}_t + (\mathbf{Q} + \mathbf{Q}')\mathbf{u}_t + \beta\mathbf{D2}'E_t\frac{\partial V(\mathbf{S}_t)}{\partial \mathbf{S}_t} = 0 \tag{107}$$

Note that since $\mathbf{Q}$ has a middle row and column of zeros, the focs will not contain any terms in the tax instrument, such that we effectively have three focs in four unknowns, π_t, y_t, g_t and $E_t \frac{\partial V(\mathbf{b}_t)}{\partial \mathbf{b}_t}$.These can be arranged as the following linear target criteria,

$$2\mathbf{L}'\mathbf{R}\boldsymbol{\pi}_t + H \begin{bmatrix} y_t \\ g_t \\ \frac{\partial V(\mathbf{b}_t)}{\partial \mathbf{b}_t} \end{bmatrix} = 0 \tag{108}$$

where $\mathbf{H} = \begin{bmatrix} 2\mathbf{Q}_{11} & \mathbf{Q}_{1,3} + \mathbf{Q}_{3,1} & \beta\mathbf{N}_{1,1} \\ \mathbf{Q}_{2,1} + \mathbf{Q}_{1,2} & \mathbf{Q}_{2,3} + \mathbf{Q}_{3,2} & \beta\mathbf{N}_{2,1} \\ \mathbf{Q}_{1,3} + \mathbf{Q}_{3,1} & 2\mathbf{Q}_{33} & \beta\mathbf{N}_{3,1} \end{bmatrix}$ and $\mathbf{Q}_{i,j}$ denotes the element contained in row i, column j of matrix $\mathbf{Q}$. This can be solved to yield the following target criteria under discretion,

$$\begin{bmatrix} y_t \\ g_t \\ E_t \frac{\partial V(\mathbf{b}_t)}{\partial \mathbf{b}_t} \end{bmatrix} = -2\mathbf{H}^{-1}\mathbf{L}'\mathbf{R}\boldsymbol{\pi}_t \equiv \mathbf{X}\pi_t \tag{109}$$

where

$$\mathbf{X} = \begin{bmatrix} -\frac{(\varphi+\sigma-(1-\beta)\frac{\bar{b}}{\bar{y}})\varepsilon}{(\varphi+\sigma)((1-\theta)+(1-\beta)\frac{\bar{b}}{\bar{y}}+1+\frac{\bar{b}}{\bar{y}}\gamma)} \\ -\frac{(\theta(\varphi+\sigma)-\sigma(1-\beta)\frac{\bar{b}}{\bar{y}})\varepsilon}{\sigma(\varphi+\sigma)((1-\theta)+(1-\beta)\frac{\bar{b}}{\bar{y}}+1+\frac{\bar{b}}{\bar{y}}\gamma)} \\ -2\frac{f1\varepsilon}{\gamma} + \frac{2\varepsilon}{(1-\theta)+(1-\beta)\frac{\bar{b}}{\bar{y}}+1+\frac{\bar{b}}{\bar{y}}\gamma)}\frac{\bar{b}}{\bar{y}} \end{bmatrix}$$

Note that the first two elements do not depend upon our 'guess' parameters,$f1..f4$, and imply that there is a linear relationship between the output gap and inflation and between the government spending gap and inflation under discretion. Using the same, plausible, restrictions on the size of the initial debt stock as under commitment above, these relationships will be negative. Therefore a shock which raises the debt stock will lead to higher inflation and lower output and government spending under discretion. Since lower output, consumption and government spending would tend to reduce marginal costs, the higher inflation must be fuelled by the tax increases required to control the government debt since taxation is distortionary and raises labour costs. This contrasts with the corresponding criteria under commitment which taxes adjust to ensure that inflation is zero, while the output gap and government spending adjust to support the new steady-state level of government debt. It is also the case that raising the degree of price flexibility (raising γ) will reduce the adjustment of output and government spending relative to inflation in responding to shocks under discretion.

The first order conditions with respect to the state variables are given by,

$$\frac{\partial V(\mathbf{S}_{t-1})}{\partial \mathbf{S}_{t-1}} = 2\mathbf{C1}'\mathbf{R}\boldsymbol{\pi}_t + \beta\mathbf{D1}'E_t\frac{\partial V(\mathbf{S}_t)}{\partial \mathbf{S}_t} \tag{110}$$

Since the state variables relating to the shock process are exogenous, we can focus on the first row of these first-order conditions to obtain,

$$\frac{\partial V(\mathbf{b}_{t-1})}{\partial \mathbf{b}_{t-1}} = 2\mathbf{C1}_{1,1}\mathbf{R}\boldsymbol{\pi}_t + \beta\mathbf{D1}_{2,1}E_t\frac{\partial V(\mathbf{b}_t)}{\partial \mathbf{b}_t} \tag{111}$$

Using the focs (i.e applying the envelope theorem),

$$\frac{\partial V(\mathbf{b}_{t-1})}{\partial \mathbf{b}_{t-1}} = \mathbf{W}\pi_t \tag{112}$$

where $\mathbf{W} \equiv [2\mathbf{C1}_{1,1}\mathbf{R} + \beta\mathbf{D1}_{2,1}\mathbf{X}_{3,1}]$.

Leading this one period and applying expectations,

$$\begin{aligned} E_t\frac{\partial V(\mathbf{b}_t)}{\partial \mathbf{b}_t} &= \mathbf{W}E_t\boldsymbol{\pi}_{t+1} \\ &= \mathbf{WFS}_t \\ &= \mathbf{WF}[\mathbf{D1S}_{t-1} + \mathbf{D2u}_t] \end{aligned} \tag{113}$$

substituting back into the focs,

$$2\mathbf{C2}'\mathbf{R}\boldsymbol{\pi}_t + (\mathbf{Q} + \mathbf{Q}')\mathbf{u}_t + \beta\mathbf{D3WF}[\mathbf{D1S}_{t-1} + \mathbf{D2u}_t] = 0 \tag{114}$$

where $\mathbf{D3} = \begin{bmatrix} \mathbf{D2}_{1,1} \\ \mathbf{D2}_{1,2} \\ \mathbf{D2}_{1,3} \end{bmatrix}$. Eliminating inflation,

$$2\mathbf{C2}'\mathbf{R}[\mathbf{C1S_{t-1}} + \mathbf{C2u_t}] + (\mathbf{Q} + \mathbf{Q}')\mathbf{u}_t + \beta\mathbf{D3WF}[\mathbf{D1S}_{t-1} + \mathbf{D2u}_t] = 0 \tag{115}$$

and solving for control variables,

$$\mathbf{u_t} = -\,[\mathbf{U1}]^{-1}\mathbf{U2S}_{t-1} \tag{116}$$

where $\mathbf{U1} = [2\mathbf{C2}'\mathbf{RC2} + [\mathbf{Q} + \mathbf{Q}'] + \beta\mathbf{D3WFD2}]$ and $\mathbf{U2} = [\mathbf{2C2}'\mathbf{RC1} + \beta\mathbf{D3WFD1}]$. The solution for inflation is now given as,

$$\boldsymbol{\pi}_t = [\mathbf{C1} - \mathbf{C2}[\mathbf{U1}]^{-\mathbf{1}}\mathbf{U2}]\mathbf{S}_{t-1} \tag{117}$$

However, this solution is a function of the undetermined coefficients, $f1$ and $f2$, which can be derived by equating coefficients,

$$\mathbf{F} = \mathbf{C1} - \mathbf{C2}[\mathbf{U1}]^{-\mathbf{1}}\mathbf{U2} \tag{118}$$

This completes the solution of the problem under discretion. Unfortunately the analytical solution defined in this way is too unwieldy to yield any real intuition. Nevertheless, after imposing the solved value for the undetermined coefficients, we can examine the evolution of the state variables under the optimal discretionary policy,

$$\mathbf{S}_t = [\mathbf{D1} - [\mathbf{U1}]^{-\mathbf{1}}\mathbf{U2}]\mathbf{S}_{t-1} \equiv \mathbf{GS}_{t-1} \tag{119}$$

In order for debt to follow a random walk under discretion in the face of non-permament shocks, the element $G_{1,1}$ must equal 1. Here we can provide a proof by counterexample- substitution of the central parameter set utilised in the simulation section below indicates that $G_{1,1} < 1$. In other words, under discretionary policy debt is eventually returned to its pre-shock level and does not automatically contain the random walk property as under the commitment case.

6 Optimal Policy Simulations

In this section we examine the optimal policy response to various shocks. We consider discretionary and commitment policies and compute the welfare benefits of employing our various fiscal instruments as stabilisation devices. In this section we outline the response of the model to a series of shocks. Following Leith and Wren-Lewis (2005) we adopt the following parameter set, $\varphi = 1$, $\mu = 1.2$, $\epsilon_t = 6$, $\theta_p = 0.75$, $\beta = 0.99$, $\alpha = 0.4$, and $1 - \theta = 0.25$. The productivity shock follows the following pattern,

$$a_t = \rho_a a_{t-1} + \xi_t \tag{120}$$

where we adopting a degree of persistence in the productivity shock of $\rho_a = 0.99$. We adopt a similar dynamic structure for the labour supply shock, consistent with the evidence for both forms of shock in Smets and Wouters (2005). Ireland (2004) finds similar persistence in the productivity shock, but lower persistence in the price mark-up shock[7] We therefore consider a cost-push shock with a similar stochastic structure but with a lower degree of persistence, $\rho_\mu = 0.9$, which is slightly lower than Ireland's estimate of 0.96. Of course, if we raise the persistence of this shock then this will raise the costs relative to the numbers we compute below.

6.1 Simulations

We then consider the ability of an economy to stabilise the economy following a productivity shock through the use of fiscal instruments when it must also ensure sustainability of the government's finances. Figure 1 details the paths of key endogenous variables following the technology shock. In the case of timeless commitment policy, we observe an instantaneous and permanent rise in taxation and fall in output and government spending, which is sufficient to support the eventual steady-state debt stock without generating inflation. This debt stock fully reflects the fiscal consequences of shocks. This is the random walk result of

[7] Smets and Wouters (2005) do not allow for persistence in the cost-push shock.

Benigno and Woodford (2003) and Schmitt-Grohe and Uribe (2004), generalised to the case of two fiscal instruments, which also has echoes of tax smoothing (Barro (1979)).

If we then allow policy makers to exploit the fact that expectations are given in the initial period, then we observe an initial rise in taxation which fuels inflation in the initial period. This actually reduces debt in the initial period, which allows lower levels of taxation and higher levels of output, government spending and consumption than would otherwise be the case for beyond the initial period. However, for this particular shock, the magnitude of these difference is small, but can be seen from the graph in the bottom right-hand cell which compares the path for debt under optimal and timeless commitment in the first two periods.

A more substantial difference occurs when we consider the discretionary solution. We demonstrated analytically that a time-inconsistency problem existed whereby negative fiscal shocks create an incentive for governments to raise taxes and, when debt is nominal, to fuel inflation to deflate the real value of the debt stock. Under discretion the national fiscal authorities, taking future inflationary expectations as given, have this incentive and economic agents incorporate this into their inflation expectations. As a consequence, the fiscal authorities are forced to raise taxes by more than they would under commitment, this fuels inflation, but also serves to reduce the debt initially. (This is in constrast to the case under commitment where the debt rises throughout the simulation). Eventually falling output reduces tax revenues, and debt slowly returns to its first-best level where the time-inconsistency problem is eliminated. The dynamics of fiscal instruments and debt depend in particular on the persistence of the shock. However, for empirically plausible measures of persistence, shocks with negative fiscal consequences will imply debt initially falling under discretion, in stark contrast to the commitment case. If we assumed a substantially less persistent shock (which is, hwoever, inconsistent with the empirical evidence), debt would initially increase, but it will always return to its steady state value under discretion, for reasons discussed above.

Similar responses emerge for other shocks present in the model since it is only to the extent that shocks have fiscal consequences that we cannot use the mixture of fiscal and monetary policy instruments to offset the impact of shocks on gap variables. These are summarised in the following table.

Table 1 - Welfare Consequences of Shocks under Alternative Policies.

	Discretion	Timeless Commitment	Commitment
Technology	0.0151%	9.5677x10^{-4}%	9.3498x10^{-4}%
Labour Supply	0.0295%	2.3919x10^{-4}%	2.3374x10^{-4}%
Mark-Up	4.2142%	0.2022%	0.1976%

The welfare implications of these shocks are limited in the case of technology shocks, with the costs of the technology shock amounting to only 9.3498x10^{-4}% of one period's steady-state consumption rate under commitment (with welfare only marginally lower in the case of timelessly optimal policy) and 0.0151 % under discretion. Labour supply shocks are similar in magnitude to technology shocks. However for autocorrelated cost-push shocks the welfare effects can be substantial under discretion - a 1% shock has a cost of 4.2142% of one period's steady-state consumption under discretion, compared with only 0.1976% (and 0.2022%) under commitment (and timelessly optimal commitment). The reason why the mark-up shock has such dramatic welfare implications is that it cannot be offset by monetary policy, but requires variations in income tax rates which have direct fiscal consequences. The other shocks can be mitigated through the use of monetary policy, which has less effect on the government's finances.

7 Conclusions

In this paper we examined the ability of fiscal and monetary instruments to offset the effects of various shocks when policymakers did not have access to lump-sum taxes to balance the budget. We found that this implied, in a simple New Keynesian framework that the three instruments available to policy makers could not offset the impact of shocks on welfare relevant gap variables since the shocks also had an impact on the government's budget constraint. We analytically derived commitment policy from the timeless perspective and found that the steady-state debt stock would fully incorporate the fiscal consequences of shocks and tax and government spending variables would only adjust to maintain this new steady-state level of debt. Inflation would be zero throughout and monetary policy would ensure that nominal interest rates were consistent with the natural rate of interest.

We then contrasted the timeless conception of commitment with the more conventional version where policy makers exploit the fact that expectations are given in the initial period. This reveals the nature of the time-inconsistency problem inherent in the commitment solution, whereby the fiscal authorities have the incentive, given expectations, to use taxes more aggressively in the initial period to reduce the subsequent debt-disequilibrium and the costs associated with sustaining a given debt level. In the case of a shock with negative fiscal consequences this will imply that the fiscal authorities aggressively raise taxes in the initial period, raising marginal costs and fueling inflation in the initial period. This slows the initial rise in the debt stock allowing the new steady-state debt stock to be supported with lower permanent increases in the debt stock, and falls in consumption, output and government spending.

We then turn to the discretionary solution, where governments follow a time consistent policy. The random walk result no longer holds, and instead debt gradually returns to its steady state level. Only by returning debt to its steady state can the incentive to reduce debt noted under commitment be eliminated. In simulations, we have shown that empirically plausible persistent shocks can lead to such aggressive tax increases that debt initially falls under discretion (compared to increasing under commitment), before returning to the steady state.

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Appendix 1 - Price Setting

Recall the optimal price set by firms that are able to reset prices in period t,

$$P_t^* = \frac{\sum_{s=0}^{\infty}(\theta_p)^s Q_{t,t+s}\left[\epsilon_t \frac{W_{t+s}}{P_{t+s}} P_{t+s}^{\epsilon_t} \frac{Y_{t+s}}{A_{t+s}}\right]}{\sum_{s=0}^{\infty}(\theta_p)^s Q_{t,t+s}\left[(\epsilon_t - 1)P_{t+s}^{-1} P_{t+s}^{\epsilon_t} Y_{t+s}(1+\chi)\right]} \tag{121}$$

Note that in equilibrium,

$$\beta^s \left(\frac{C_t}{C_{t+s}}\right)\left(\frac{P_t}{P_{t+s}}\right) = Q_{t,t+s} \tag{122}$$

Accordingly, the expression for the optimal price can be re-written as,

$$P_t^* = \frac{\sum_{s=0}^{\infty}(\theta_p\beta)^s \frac{C_t P_t}{C_{t+s} Pt+s}\left[\epsilon_t \frac{W_{t+s}}{P_{t+s}} P_{t+s}^{\epsilon_t} \frac{Y_{t+s}}{A_{t+s}}\right]}{\sum_{s=0}^{\infty}(\theta_p\beta)^s \frac{C_t P_t}{C_{t+s} Pt+s}\left[(\epsilon_t - 1)P_{t+s}^{-1} P_{t+s}^{\epsilon_t} Y_{t+s}(1+\chi)\right]} \tag{123}$$

This can be loglinearised as,

$$p_t^* = (1-\theta_p\beta)E_t\left(\sum_{s=0}^{\infty}(\theta_w\beta)^s\left[-a_{t+s} + w_{t+s} - v_t + \ln(\mu_t)\right]\right) \tag{124}$$

where p_t^* is the log of the optimal price set by those firms that were able to set price in period t, $\ln(\mu_t) = \frac{\epsilon_t}{\epsilon_t - 1}$ is the desired mark-up and $v = -\ln(1+\chi)$.Quasi-differencing this expression yields,

$$\frac{1}{1-\theta_p\beta}p_t^* = \frac{1}{1-\theta_p\beta}\theta_p\beta E_t p_{t+1}^* - a_t + w_t - v_t + \ln(\mu_t) \tag{125}$$

While domestic prices evolve according to,

$$P_t = \left[(1-\theta_p)P_t^{*(1-\epsilon_t)} + \theta_p P_{t-1}^{1-\epsilon_t}\right]^{\frac{1}{1-\epsilon_t}} \tag{126}$$

This can be log-linearised as,

$$p_t = (1-\theta_p)p_t^* + \theta_p p_{t-1} \tag{127}$$

Solving for p_t^* and substituting into the expression for quasi-differenced optimal price yields,

$$\begin{aligned}\frac{1}{1-\theta_p\beta}\left(\frac{p_t}{1-\theta_p} - \frac{\theta_p p_{t-1}}{1-\theta_p}\right) &= \frac{1}{1-\theta_p\beta}\theta_p\beta\left(\frac{E_t p_{t+1}}{1-\theta_p} - \frac{\theta_p p_t}{1-\theta_p}\right) \\ &\quad -a_t + w_t - v_t + \ln(\mu_t)\end{aligned} \tag{128}$$

This can be solved as,

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\theta_p\beta)(1-\theta_p)}{\theta_p}(-a_t + w_t - p_{H,t} - v_t + \ln(\mu_t)) \tag{129}$$

where $mc = -a_t + w_t - p_{H,t} - v_t$ are the real log-linearised marginal costs of production. In the absence of sticky prices profit maximising behaviour implies, $mc_t = -\ln(\mu_t)$.

Appendix 3 - Derivation of Welfare

In order to illustrate how this model can be utilised to generate utility-based welfare measures to guide the optimal setting of policy we simplify the model by ignoring wage-stickiness and adopting a competitive labour market. (Reintroducing sticky-wages to the derivation of welfare is undertaken below). Individual utility in period t is

$$\frac{C_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma}+\chi\frac{G_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma}-\frac{N_t{}^{1+\varphi}\xi_t^{-\sigma}\xi_t^{N}}{1+\varphi} \tag{130}$$

Before considering the elements of the utility function we need to note the following general result relating to second order approximations,

$$\frac{Y_t-Y}{Y_t}=\widehat{Y}_t+\frac{1}{2}\widehat{Y}_t^2+O[3] \tag{131}$$

where $\widehat{Y}_t=\ln(\frac{Y_t}{Y})$, $O[3]$ represents terms that are of order higher than 3 in the bound on the amplitude of the relevant shocks.

Suppose we take the Taylor series expansion of $\ln(\frac{Y_t}{Y})$, we obtain,

$$\ln(\frac{Y_t}{Y})=\ln(1)+\frac{1}{Y}(Y_t-Y)-\frac{1}{2}\frac{1}{Y^2}(Y_t-Y)^2+O[3] \tag{132}$$

Soving for the precentage deviation we have,

$$\frac{1}{Y}(Y_t-Y)=\ln(\frac{Y_t}{Y})+\frac{1}{2}\frac{1}{Y^2}(Y_t-Y)^2+O[3] \tag{133}$$

Since we are ignoring all terms of order higher than 2 we can rewrite the second order term as follows,

$$\frac{1}{Y}(Y_t-Y)=\ln(\frac{Y_t}{Y})+\frac{1}{2}\ln(\frac{Y_t}{Y})^2+O[3] \tag{134}$$

This will be used in various places in the derivation of welfare. Now consider the second order approximation to the first term,

$$\begin{aligned}\frac{C_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma} &= \frac{\overline{C}^{1-\sigma}}{1-\sigma}+\overline{C}^{-\sigma}(C_t-\overline{C})-\frac{\sigma}{2}\overline{C}^{-\sigma-1}(C_t-\overline{C})^2 \\ &\quad -\sigma\overline{C}^{-\sigma}(C_t-\overline{C})(\xi_t-1)-\frac{\sigma}{2}\sigma\overline{C}^{1-\sigma}(\xi_t-1)^2+O[3]\end{aligned} \tag{135}$$

This can be rewritten as,

$$\begin{aligned}\frac{C_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma} &= \overline{C}^{1-\sigma}(\frac{C_t-\overline{C}}{\overline{C}})-\frac{\sigma}{2}\overline{C}^{1-\sigma}(\frac{C_t-\overline{C}}{\overline{C}})^2 \\ &\quad -\sigma\overline{C}^{1-\sigma}(\frac{C_t-\overline{C}}{\overline{C}})(\xi_t-1)+tip+O[3]\end{aligned} \tag{136}$$

Using the results above this can be rewritten in terms of hatted variables,

$$\frac{C_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma} = \overline{C}^{1-\sigma}\{\widehat{C}_t + \frac{1}{2}(1-\sigma)\widehat{C}_t^2 - \sigma\widehat{C}_t\widehat{\xi}_t\} + tip + O[3] \quad (137)$$

Similarly for the term in government spending,

$$\chi\frac{G_t^{1-\sigma}\xi_t^{-\sigma}}{1-\sigma} = \chi\overline{G}^{1-\sigma}\{\widehat{G}_t + \frac{1}{2}(1-\sigma)\widehat{G}_t^2 - \sigma\widehat{G}_t\widehat{\xi}_t\} + tip + O[3] \quad (138)$$

The final term in labour supply can be written as,

$$\frac{N_t^{1+\varphi}\xi_t^N\xi_t^{-\sigma}}{1+\varphi} = \overline{N}^{1+\varphi}\{\widehat{N}_t + \frac{1}{2}(1+\varphi)\widehat{N}_t^2 - \sigma\widehat{N}_t\widehat{\xi}_t + \widehat{N}_t\widehat{\xi}_t^N\} + tip + O[3] \quad (139)$$

Now we need to relate the labour input to output and a measure of price dispersion. Aggregating the individual firms' demand for labour yields,

$$N = (\frac{Y}{A})\int_0^1 (\frac{P_H(i)}{P_H})^{-\epsilon_t}di \quad (140)$$

It can be shown that

$$\widehat{N} = \widehat{Y} - a + \ln[\int_0^1 (\frac{P(i)}{P})^{-\epsilon_t}di] \quad (141)$$

$$= \widehat{Y} - a + \frac{\epsilon_t}{2}var_i\{p(i)\} + O[3] \quad (142)$$

In other words, a wider dispersion of prices means that more workers are employed to produce a given level of aggregate output/consumption. While the first line is straight-forward, there are several steps behind the next line. The first things to note, following Woodford (2003, Chapter 6) is that $z_t = \ln[\int_0^1 (\frac{P(i)}{P})^{-\epsilon_t}di]$ is of second order.

$$z_t = \frac{\varepsilon}{2}var_i\{p_{H,t}(i)\} + O[3]$$

so we can write

$$\frac{N_t^{1+\varphi}\xi_t^N\xi_t^{-\sigma}}{1+\varphi} = \overline{N}^{1+\varphi}\{\widehat{Y}_t + \frac{1}{2}(1+\varphi)\widehat{Y}_t^2 - (1+\varphi)\widehat{Y}_t a_t + \widehat{Y}_t\widehat{\xi}_t^N - \sigma\widehat{Y}_t\widehat{\xi}_t + \frac{\epsilon_t}{2}var_i\{p_t(i)\}\} + tip + O[3] \quad (143)$$

Now consider the dispersion measure, $var_i\{p_t(i)\}$ for which Woodford (2003, Chapter 6) demonstrates,

$$\sum_{t=0}^{\infty}\beta^t var_i\{p_t(i)\} = \sum_{t=0}^{\infty}\beta^t\left(\theta_p^{t+1}var_i\{p_{-1}(i)\} + \sum_{s=0}^{t}\frac{\theta_p}{1-\theta_p}\pi_s^2 + o\left(\|a\|^3\right)\right) + O[3]$$

$$= \frac{\theta_p}{(1-\theta_p)(1-\theta_p\beta)}\sum_{t=0}^{\infty}\beta^t\pi_t^2 + t.i.p + O[3]$$

Using these expansions, individual utility can be written as

$$\begin{aligned}\Gamma_t &= \overline{C}^{1-\sigma}\{\widehat{C}_t + \frac{1}{2}(1-\sigma)\widehat{C}_t^2 - \sigma\widehat{C}_t\widehat{\xi}_t\} \\ &+\chi\overline{G}^{1-\sigma}\{\widehat{G}_t + \frac{1}{2}(1-\sigma)\widehat{G}_t^2 - \sigma\widehat{G}_t\widehat{\xi}_t\} \\ &-\overline{N}^{1+\varphi}\{\widehat{Y}_t + \frac{1}{2}(1+\varphi)\widehat{Y}_t^2 - (1+\varphi)\widehat{Y}_t a_t \\ &+\widehat{Y}_t\widehat{\xi}_t^N + \frac{\epsilon_t}{2}var_i\{p_t(i)\}\} \\ &+tip + O[3]\end{aligned} \tag{144}$$

Using second order approximation to the national accounting identity,

$$\theta\widehat{C}_t = \widehat{Y}_t - (1-\theta)\widehat{G}_t - \frac{1}{2}\theta\widehat{C}_t^2 - \frac{1}{2}(1-\theta)\widehat{G}_t^2 + \frac{1}{2}\widehat{Y}_t^2 + O[3] \tag{145}$$

With the steady-state subsidy in place and government spending chosen optimally, the following conditions hold in the initial steady-state, $\overline{C}^{1-\sigma}$

$$\overline{C}^{1-\sigma} = \overline{N}^{1+\varphi}\theta \tag{146}$$

and,

$$\chi\overline{G}^{1-\sigma} = \overline{N}^{1+\varphi}(1-\theta) \tag{147}$$

Which allows us to eliminate the levels terms and rewrite welfare as,

$$\begin{aligned}\Gamma_t &= \overline{C}^{1-\sigma}\{-\frac{1}{2}\sigma\widehat{C}_t^2 - \sigma\widehat{C}_t\widehat{\xi}_t\} + \chi\overline{G}^{1-\sigma}\{-\frac{1}{2}\sigma\widehat{G}_t^2 - \sigma\widehat{G}_t\widehat{\xi}_t\} \\ &-\overline{N}^{1+\varphi}\{\frac{1}{2}\varphi\widehat{Y}_t^2 - (1+\varphi)\widehat{Y}_t a_t + \widehat{Y}_t\widehat{\xi}_t^N + \frac{\epsilon_t}{2}var_i\{p_t(i)\}\} \\ &+tip + O[3]\end{aligned} \tag{148}$$

We now need to rewrite this in gap form. To do this consider the focs for the social planner,

$$(Y_t - G_t)^{-\sigma}\xi_t^{-\sigma} - Y_t^{\varphi}A_t^{-(1+\varphi)}\xi_t^{\varphi} = 0 \tag{149}$$

and for G,

$$-(Y_t - G_t)^{-\sigma}\xi_t^{-\sigma} + \chi G_t{}^{-\sigma}\xi_t^{-\sigma} = 0 \tag{150}$$

Log-linearising these,

$$-\sigma\widehat{C}_t^* - \sigma\widehat{\xi}_t = \varphi\widehat{Y}_t^* - (1+\varphi)a_t + \varphi\widehat{\xi}_t \tag{151}$$

and,

$$\widehat{C}_t^* = \widehat{G}_t^* \tag{152}$$

where the '*' denotes the efficient level of that variable in the face of shocks. From the national accounting identity the latter implies, $\widehat{C}_t^* = \widehat{G}_t^* = \widehat{Y}_t^*$. Using the definition of the optimal level of output in the face of shocks we can eliminate the cross term in the technology shock,

$$\begin{aligned}\Gamma_t &= \overline{C}^{1-\sigma}\{-\frac{1}{2}\sigma\widehat{C}_t^2 - \sigma\widehat{C}_t\widehat{\xi}_t\} + \chi\overline{G}^{1-\sigma}\{-\frac{1}{2}\sigma\widehat{G}_t^2 - \sigma\widehat{G}_t\widehat{\xi}_t\} \\ &\quad -\overline{N}^{1+\varphi}\{\frac{1}{2}\varphi\widehat{Y}_t^2 - \varphi\widehat{Y}_t^*\widehat{Y}_t - \sigma\widehat{C}_t^*\widehat{Y}_t - \sigma\widehat{\xi}_t\widehat{Y}_t + \frac{\epsilon_t}{2}var_i\{p_t(i)\}\} \\ &\quad +tip + O[3]\end{aligned} \tag{153}$$

Since, $\widehat{C}_t^* = \widehat{G}_t^* = \widehat{Y}_t^*$, the cross terms in shocks all cancel, leaving,

$$\begin{aligned}\Gamma_t &= \overline{C}^{1-\sigma}\{-\frac{1}{2}\sigma(\widehat{C}_t - \widehat{C}_t^*)^2\} + \chi\overline{G}^{1-\sigma}\{-\frac{1}{2}\sigma(\widehat{G}_t - \widehat{G}_t^*)^2\} \\ &\quad -\overline{N}^{1+\varphi}\{\frac{1}{2}\varphi(\widehat{Y}_t - \widehat{Y}_t^*)^2 - \sigma\widehat{C}_t^*\widehat{Y}_t + \frac{\epsilon_t}{2}var_i\{p_t(i)\}\} \\ &\quad -\sigma\overline{C}^{1-\sigma}\widehat{C}_t\widehat{C}_t^* - \sigma\chi\overline{G}^{1-\sigma}(\widehat{G}_t - \widehat{G}_t^*)^2 \\ &\quad +tip + O[3]\end{aligned} \tag{154}$$

Which, given the efficient initial steady-state simplifies to,

$$\begin{aligned}\Gamma_t &= \overline{C}^{1-\sigma}\{-\frac{1}{2}\sigma(\widehat{C}_t - \widehat{C}_t^*)^2\} + \chi\overline{G}^{1-\sigma}\{-\frac{1}{2}\sigma(\widehat{G}_t - \widehat{G}_t^*)^2\} \\ &\quad -\overline{N}^{1+\varphi}\{\frac{1}{2}\varphi(\widehat{Y}_t - \widehat{Y}_t^*)^2 - \sigma\widehat{C}_t^*\widehat{Y}_t + \frac{\epsilon_t}{2}var_i\{p_t(i)\}\} \\ &\quad -\sigma\overline{C}^{1-\sigma}\widehat{C}_t\widehat{C}_t^* - \sigma\chi\overline{G}^{1-\sigma}(\widehat{G}_t - \widehat{G}_t^*)^2 \\ &\quad +tip + O[3]\end{aligned} \tag{155}$$

Rewriting in gap form,

$$\begin{aligned}\Gamma_t &= -\overline{N}^{1+\varphi}\frac{1}{2}\{\sigma\theta(\widehat{C}_t - \widehat{C}_t^*)^2 + \sigma(1-\theta)(\widehat{G}_t - \widehat{G}_t^*)^2 + \varphi(\widehat{Y}_t - \widehat{Y}_t^*)^2 + \epsilon_t var_i\{p_t(i)\}\} \\ &\quad +tip + O[3]\end{aligned} \tag{156}$$

Using the result from Woodford (2003) that

$$\sum_{t=0}^{\infty}\beta^t var_i\{p_t(i)\} = \frac{\theta}{(1-\theta)(1-\theta\beta)}\sum_{t=0}^{\infty}\beta^t\pi_t^2 + t.i.p + O[3] \tag{157}$$

we can write the discounted sum of utility as,

$$\begin{aligned}\Gamma = -\overline{N}^{1+\varphi}\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\{\sigma\theta(\widehat{C}_t - \widehat{C}_t^*)^2 &\quad +\sigma(1-\theta)(\widehat{G}_t - \widehat{G}_t^*)^2 + \varphi(\widehat{Y}_t - \widehat{Y}_t^*)^2 + \frac{\epsilon_t}{\gamma}\pi_t^2\} \\ &\quad +tip + O[3]\end{aligned} \tag{158}$$

Welfare is the sum of quadratic terms in inflation, the output, consumption and government spending gaps. Optimal monetary and fiscal policy will maximise this subject to the constraints implied by the three dynamic equations for each country, in output, inflation and government debt.

Appendix 4

The log-linearised budget constraint is given by,

$$\begin{aligned} \widehat{b}_{t-1} - \pi_t &= \beta E_t\{\widehat{b}_t - \pi_{t+1}\} \\ &+ [\frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}(\widehat{w}_t + \widehat{N}_t + \widehat{\tau}_t) - \frac{\overline{G}}{\overline{b}}\widehat{G}_t] \end{aligned} \tag{159}$$

From labour supply,

$$-\frac{\overline{\tau}}{1-\overline{\tau}}\widehat{\tau}_t + \widehat{w}_t = \varphi\widehat{N}_t + \sigma\widehat{C}_t + \widehat{\xi}_t^N \tag{160}$$

Eliminating real wages from the budget constraint,

$$\begin{aligned} \widehat{b}_{t-1} - \pi_t &= \beta E_t\{\widehat{b}_t - \pi_{t+1}\} \\ &+ [\frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}((1+\varphi)\widehat{Y}_t - (1+\varphi)a_t + \frac{1}{1-\overline{\tau}}\widehat{\tau}_t + \sigma\widehat{C}_t + \widehat{\xi}_t^N) \\ &- \frac{\overline{G}}{\overline{b}}\widehat{G}_t] \end{aligned} \tag{161}$$

Using the definition of efficient output,

$$-\sigma\widehat{C}_t^* = \varphi\widehat{Y}_t^* - (1+\varphi)a_t + \widehat{\xi}_t^N \tag{162}$$

to eliminate the term in the technology shock,

$$\begin{aligned} \widehat{b}_{t-1} - \pi_t &= \beta E_t\{\widehat{b}_t - \pi_{t+1}\} \\ &+ \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}((1+\varphi)(\widehat{Y}_t - \widehat{Y}_t^*) + \frac{1}{1-\overline{\tau}}\widehat{\tau}_t \\ &+ \sigma(\widehat{C}_t - \widehat{C}_t^*) + \widehat{Y}_t^*) \\ &- \frac{\overline{G}}{\overline{b}}\widehat{G}_t] \end{aligned} \tag{163}$$

Gapping the remaining variables,

$$\begin{aligned} \widehat{b}_{t-1} - \pi_t &= \beta E_t\{\widehat{b}_t - \pi_{t+1}\} \\ &+ \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}((1+\varphi)(\widehat{Y}_t - \widehat{Y}_t^*) + \frac{1}{1-\overline{\tau}}\widehat{\tau}_t + \sigma(\widehat{C}_t - \widehat{C}_t^*)) \\ &- \frac{\overline{G}}{\overline{b}}(\widehat{G}_t - \widehat{G}_t^*) + \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}\widehat{Y}_t^* - \frac{\overline{G}}{\overline{b}}\widehat{G}_t^* \end{aligned} \tag{164}$$

Recall that, $\widehat{C}_t^* = \widehat{G}_t^* = \widehat{Y}_t^*$, allows us to rewrite this as,

$$\begin{aligned} \widehat{b}_{t-1} - \pi_t &= \beta E_t\{\widehat{b}_t - \pi_{t+1}\} - \frac{\overline{G}}{\overline{b}}(\widehat{G}_t - \widehat{G}_t^*) + \widehat{Y}_t^*[\frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}} - \frac{\overline{G}}{\overline{b}}] \\ &+ \frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}((1+\varphi)(\widehat{Y}_t - \widehat{Y}_t^*) + \frac{1}{1-\overline{\tau}}\widehat{\tau}_t + \sigma(\widehat{C}_t - \widehat{C}_t^*)) \end{aligned} \tag{165}$$

Simplifying,

$$\begin{aligned}\widehat{b}_{t-1}-\pi_t &= \beta E_t\{\widehat{b}_t-\pi_{t+1}\}-\frac{\overline{G}}{\overline{b}}(\widehat{G}_t-\widehat{G}_t^*) \\ &+\frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}((1+\varphi)(\widehat{Y}_t-\widehat{Y}_t^*)+\frac{1}{1-\overline{\tau}}\widehat{\tau}_t+\sigma(\widehat{C}_t-\widehat{C}_t^*)) \\ &+\widehat{Y}_t^*[(1-\beta)]\end{aligned} \tag{166}$$

Combining shock terms,

$$\begin{aligned}\widehat{b}_{t-1}-\pi_t &= \beta E_t\{\widehat{b}_t-\pi_{t+1}\}-\frac{\overline{G}}{\overline{b}}(\widehat{G}_t-\widehat{G}_t^*) \\ &+\frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}[(1+\varphi)(\widehat{Y}_t-\widehat{Y}_t^*)+\frac{1}{1-\overline{\tau}}(\widehat{\tau}_t-\widehat{\tau}_t^*)+\sigma(\widehat{C}_t-\widehat{C}_t^*)]-f_t\end{aligned} \tag{167}$$

where $f_t=-\widehat{Y}_t^*(1-\beta)-\frac{\overline{w}\overline{N}\overline{\tau}}{\overline{b}}\frac{1}{1-\overline{\tau}}\widehat{\tau}_t^*$ is a measure of the fiscal consequences of shocks. Using the definition of efficient output,

$$\widehat{Y}_t^*=\frac{(1+\varphi)}{\sigma+\varphi}a_t-\frac{\widehat{\xi}_t^N}{\sigma+\varphi} \tag{168}$$

we can rewrite the fiscal variable,

$$f_t=-(1-\beta)\frac{(1+\varphi)}{\sigma+\varphi}a_t+\frac{(1-\beta)}{\sigma+\varphi}\widehat{\xi}_t^N-\frac{\overline{w}\overline{N}}{\overline{b}}\mu_t \tag{169}$$

as a function of the various shocks hitting the economy which have fiscal consequences.

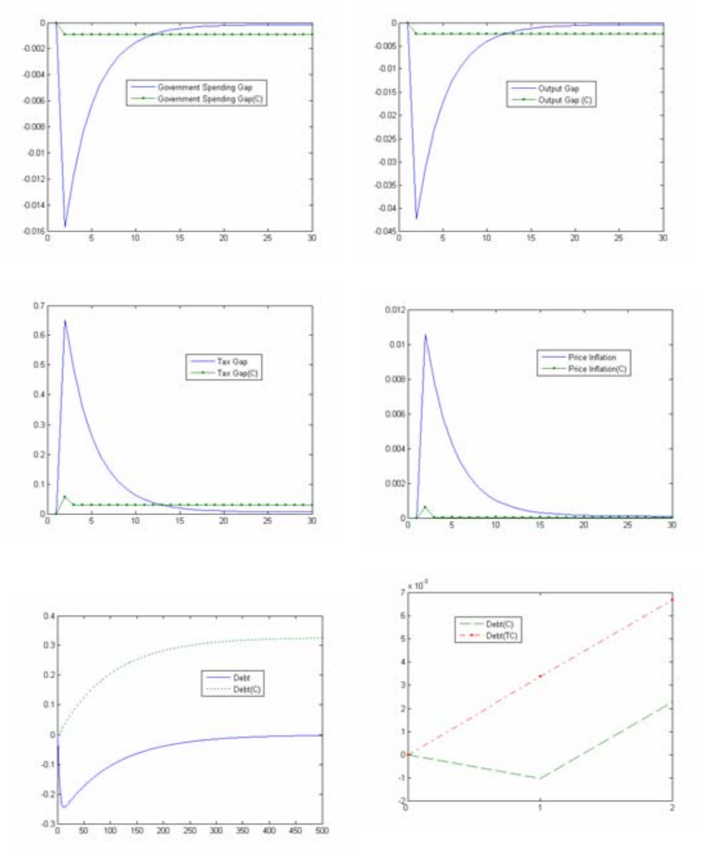

Figure 1: Response of Endogenous Variables to a 1% Technology Shock

Notes to Figure: Time period in graph in bottom left is 500 periods rather than 30, while the graph in the bottom right cell gives the paths for debt for the first two periods under optimal and timeless commitment policy, respectively.