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The Road to Extinction: Commons with Capital Markets

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The road to extinction: commons with capital markets[1]

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Abstract

We study extinction in a commons problem in which agents have access to capital markets. When the commons grows more quickly than the interest rate, multiple equilibria are found for intermediate commons endowments. In one of these, welfare decreases as the resource becomes more abundant, a 'resource curse'. As marginal extraction costs become constant, market access instantly depletes the commons. Without markets - the classic environment - equilibria are unique; extinction dates and welfare increase with the endowment. When the endowment is either very abundant or very scarce, market access improves welfare. As marginal costs of extraction from the commons become constant, market access can reduce welfare if the subjective discount rate exceeds the interest rate.

Key words: commons, capital markets, perfect foresight, extinction, resource curse, storage

JEL classification numbers: C73, D91, O17, Q21

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1 Introduction

We analyse a dynamic commons problem in which agents have access to capital markets in order to address two questions. First, how does capital market access alter behaviour relative to that under intertemporal autarky? Second, what are the welfare consequences of capital market access?

Theoretically, the question is a natural one: commons problems and capital markets, two basic objects of economic theory, have largely been studied in isolation. The welfare properties of markets are well known; the commons is an equally well known example of market failure. Understanding better the interaction between these objects therefore seems promising.

Practically, all but the poorest agents in a contemporary commons problem are likely to have access to some form of capital market. To the extent that this changes their behaviour, and that of the commons, it is therefore important to understand the consequences of such access. To illustrate, we mention two motivating examples.

The first owes to Tornell and Velasco (1992), who asked why capital could often be observed flowing from poor countries, where it was scarce, to rich ones.[1] Noting that property rights may be weak in the poor country, they modelled the extreme case, in which capital stocks were actually communal endowments. Domestically, scarce capital earned a high return, but one to which its 'owners' had no greater rights than other members of society. Capital could be extracted from this commons and sent abroad, endowing private savings accounts which earned lower, but secure rates of return.[2]

In the second example, the commons is the atmosphere, with extraction from it corresponding to its qualitative rather than quantitative degradation. Interaction with capital markets occurs when producers earn profits in part by emitting into the atmosphere, or borrow against a plan to emit in the future.

While the first best in both cases may involve both strengthening property rights and enabling capital market access, it may not always be feasible to do both: strengthening property rights, which requires the extension of

[1]Gourinchas and Jeanne (2006) argue that Africa is actually capital abundant when human capital and domestic distortions are taken into account. They find that the benefits of financial integration to a typical non-OECD country are roughly equivalent to a 1% permanent increase in consumption, similar in magnitude to the costs of increased volatility.

[2]In a companion paper, Tornell and Lane (1999) interest groups compete for government revenues. Thus, the formal sector earns a high return, but is converted into a commons by government's coercive taxation power; the informal sector earns low but private returns, escaping taxation. More exotically, Ross (2003) discusses 'booty futures' in the context of civil wars.

governance, may be more difficult than easing market access, which involves loosening state controls.

Simple calculations support this concern. In terms of the first example, we use the regulatory quality (RQ) and rule of law (RL) measures from Kaufmann, Kraay, and Mastruzzi (2005) as rough proxies for liberalisation and the strength of property rights. Countries with 1996 RQ scores, the first year available, in the bottom half of the sample improved by an average of 0.03 to 2004, the last year available. For RL, the corresponding figure is -0.10, a deterioration.[3]

Thus, a poor country encouraged to undertake 'Washington Consensus' reforms, or a country suffering from a collapse in governance or regulation, may find itself in a transition period during which liberalised markets exist alongside weak property rights.[4] This may aid capital flight from a country with a high intrinsic return to capital.[5]

As to the second example, while technology developed since the Industrial Revolution has allowed substantial increases in atmospheric greenhouse gas stocks, attempts to regulate their emission are still in their infancy.

To understand these issues, Section 2 presents an infinite horizon model that generalises those in Tornell and Velasco (1992) and Tornell and Lane (1999) in three ways. First, it allows extraction to be costly. Second, extraction rates are not restricted to be shares of the communal endowment. This assumption, made for tractability's sake in the Tornell papers, rules out the possibility of exhausting the commons. Third, the commons' growth rate may be less than the interest rate, allowing consideration of non-renewable communal resources.

As a first step towards strategic analysis, we consider a continuum of identical, competitive agents who take the commons' extinction date as given. In a model with two time periods, Rowat and Dutta (forthcoming) show that equilibria with strategic agents who consider their effect on the extinction date tend to competitive equilibria as the number of agents becomes large;

[3]As both measures have zero mean in every year, comparability is a question if global averages vary over time. Kaufmann et al. (2005, §2.4) "cautiously" conclude that they "do not have any evidence of any significant improvement in governance worldwide, and if anything the evidence is suggestive of a deterioration, at the very least in key dimensions such as regulatory quality, rule of law, and control of corruption."

[4]Williamson (2000) provides a more detailed discussion of the use - and misuse - of the term 'Washington Consensus'. The widespread and ongoing looting in Iraq since 2003 is a striking example of the latter case.

[5]Prasad, Rogoff, Wei, and Kose (2003) provides an empirical discussion of liberalisation. One concern raised is that financial integration is associated with increased consumption volatility for many developing countries.

the strategic equilibria dominate the competitive ones.[6]

This, then, is the first paper to consider competitive extraction from the commons in an infinite horizon model in which agents have access to capital markets.

Our first main result, in Section 3, characterises equilibrium extinction dates. A scarce communal endowment yields a unique, finite extinction date. At the opposite extreme, with an abundant endowment, the commons is never depleted. In an intermediate range, two finite extinction dates coexist alongside the non-extinction equilibrium. The finite extinction dates reflect strategic complementarities in extraction. This reasoning only holds when the commons grows more quickly than do privately saved resources. Otherwise, there is no incentive to conserve the commons: agents will not reduce their own extraction in response to delayed extinction.

The equilibria are Pareto ranked, with later extinction dates preferred. As the communal endowment grows, the two finite extinction dates converge from opposite directions. The equilibrium with later extinction seems to suffer from a form of resource curse, whereby increases in communal endowment trigger a more intensive struggle for resources, reducing welfare.

Multiple equilibria are found elsewhere in the related literature. In Rowat and Dutta (forthcoming), multiple extinction dates may occur with market access under both strategic and competitive equilibrium concepts; under intertemporal autarky, extinction dates are unique, as here.

The recent 'storage' literature, which allows agents to save but not to borrow, also exhibit multiple equilibria. In Kremer and Morcom (2000), the ability to privately store communal resources at the opportunity cost introduces strategic complementarities.[7]

Our second result is that, in the limit as marginal extraction costs become constant, extinction becomes immediate. This result is consistent with the 'jump extinctions' of underground oil reserves, annual fishing quotas, groundwater and currency pegs studied by Gaudet, Moreaux, and Salant

[6] Tornell and Velasco (1992), on the other hand, find that increasing the number of agents increases the societal growth rate. However, in addition to the restrictions mentioned above, Tornell and Velasco (1992) and Tornell and Lane (1999) do not recognise that even Markov perfect strategies will generally depend on agents' accumulated private savings or debts. Instead, their agents condition behaviour only on the communal endowment. Braguinsky and Myerson (2006) also consider strategic oligarchs who choose to allocate assets between domestic and foreign holdings; theirs do not have access to capital markets, nor is there a communal domestic capital stock to pillage.

[7] In Homans and Wilen (2005), fish caught can either be sold immediately on the premium fresh fish market, or stored for the lower value frozen fish market. They argue that increased rents both induce entry and shorten the fishing season, thus causing more fish to be sold on the inferior market.

(2002). Considering competitive equilibria in a commons environment with private storage (rather than full capital market access), they found sudden depletions as average extraction costs became constant.[8]

These results make the restrictions on extraction in Tornell and Velasco (1992) and Tornell and Lane (1999) very restrictive: as their costless extraction implies constant marginal costs, instant extraction might be expected in the absence of other assumptions. For example, while Long and Sorger (2006) model linear extraction costs, they avoid extinction by granting agents utility from the commons stock itself with infinite marginal utility as the stock goes to zero. While their agents may only save they earn interest, unlike in the storage literature, when so doing.

Section 4 characterises behaviour without market access. This intertemporally autarkic environment is the standard dynamic commons, and has been extensively studied (q.v. Mirman, 1979; Levhari and Mirman, 1980; Benhabib and Radner, 1992; Dutta and Sundaram, 1993; Dockner and Sorger, 1996; Sorger, 1998). As the commons is now the unique source of the consumption good, agents may have more incentives to conserve it than they would with market access.[9] Extinction dates are unique: scarce communal endowments will be exhausted in finite time; abundant ones will not be exhausted. This owes to agents extracting as if they were solving a static problem: not regarding themselves as responsible for the extinction date, they view the problem as stationary until extinction. Altering the extinction date does not, therefore, induce an extraction rate response.

Our third result, then, compares welfare under market access to that under intertemporal autarky. To do so, we cannot adopt the usual approach of dissipating all rents by free entry (q.v. Kremer and Morcom, 2000; Gaudet et al., 2002). Instead, while we model a continuum of agents, we do not allow its cardinality to increase. Our main findings are as follows.

When the communal endowment is abundant (so that it would not be extinguished under either market access or autarky), market access is preferred: as agents only interact via the extinction date, setting it to infinity removes their interaction. In this circumstance, market access replaces instantaneous budget constraints with a single intertemporal one. When the resource is

[8]In Rowat and Dutta (forthcoming), the commons may survive even with constant, but positive, marginal costs: agents may reach their glut points with only two consumption periods.

[9]The 'tragedy' is that overly rapid exploitation is still typical. Exceptions can be found in Dutta and Sundaram (1993) in which, drawing on an idea from Fudenberg and Tirole (1983), agents who define trigger strategies on the state variable exploit the resource inefficiently slowly. Benhabib and Radner (1992); Dockner and Sorger (1996); Sorger (1998); Rowat (forthcoming) derive conditions under which equilibria are efficient.

scarce, market access introduces a strategic interaction, which may reduce welfare.

Market access is also preferred to autarky when the endowment is scarce and consumption utility is strongly concave, earning infinite disutility once extinction occurs and consumption ceases. When consumption utility is less concave, so that zero consumption earns finite utility, market access is preferred to autarky as the resource becomes increasingly scarce: depletion occurs quickly under both institutions, but market access allows consumption benefits long after the resource is exhausted.

In the limit case of constant marginal extraction costs, low interest rates favour autarky. As mentioned above, market access allows 'jump extinctions', instantly depleting the communal endowment and endowing private accounts. When the interest rate is low, though, impatient agents do not benefit much from this. Autarky prevents such jump extinctions: extraction beyond agents' static glut points is wasted. Thus, in the absence of strong property rights, autarky provides a second best commitment technology to preserve the high return commons.

When, on the other hand, marginal consumption utility becomes constant, market access leads agents to instantly consume their whole endowment, paying off their accrued debts over time. While autarkic consumption is constant until it depletes the commons, this cannot be a reason for preferring autarky: as there are no externalities on the consumption side of the problem, consumption plans (conditional on intertemporal extraction) must be better under market access. Autarky can outperform market access when the communal endowment is abundant and agents' subjective discount rates slightly exceed the interest rate.[10] In these cases, the negative effect of market access when the return to the commons exceeds that to secure private savings - accelerated extinction of the commons - dominates its positive effect - superior intertemporal smoothing. As impatience grows, the extinction date further reduces; this negative effect is outweighed by the positive effect of the reduced present value of extraction costs.

In both of the limit cases mentioned, agents face only one smoothing problem. Otherwise, with strictly concave consumption utility and strictly convex extraction costs, agents face both consumption and extraction smoothing problems. Under market access, they have two instruments with which to address these. Under autarky, with instantaneous consumption constrained by instantaneous extraction, they only have one. The limit cases mentioned are therefore the sorts of cases in which one would expect autarky to have

[10]In Rowat and Dutta (forthcoming), low interest rates relative to subjective discount rates also favour autarky.

the smallest disadvantages.

Section 5 concludes. The Appendix collects the main proofs.

2 The model

Time, indexed by t, runs continuously to infinity. Agents, indexed by i, are identical, infinitesimal, and uniformly distributed on the unit interval. Society is communally endowed with a single consumption good, $k(0)$.[11] Its instantaneous growth rate is the constant, a. At each instant, each agent extracts $x_i(t) \geq 0$ from the commons, and consumes $c_i(t) \geq 0$ of the consumption good. Thus, $x_i(t)$ is the (finite) extraction rate at time t of infinitesimal agent i.

Aggregate extraction is $x(t) = \int x_i(t)\, di$. Under symmetric extraction, $x(t) = x_i(t)$. The aggregate extraction path, $x(t)$, defines the initial value problem

$$\dot{k}(t) = ak(t) - x(t), k(0) > 0 \tag{1}$$

whose solution is the path of the commons stock whenever $k(t) > 0$.

An *extinction date* is the smallest $T \geq 0$, such that $k(T) = 0$. If $\lim_{t\to\infty} k(t) > 0$, then $T = \infty$, which corresponds to non-extinction. Feasible extraction cannot exceed the stock:

$$k(t) = 0 \Rightarrow x_i(t) = 0. \tag{2}$$

For all $t \geq T$, the commons stock is therefore absorbed at $k(t) = 0$, so that

$$k(t) = \max\left\{0, k(0)\, e^{at} - \int_0^t e^{a(t-\tau)} x(\tau)\, d\tau\right\}. \tag{3}$$

Equation 3 describes the unique solution to initial value problem whenever $x(t)$ is continuous over $[0, T)$ (Walter, 1998, p.28). Thus, aggregate extraction is *admissible* if $x(t)$ is continuous over $[0, T)$.[12]

Individuals choose extraction and consumption paths to maximise utility,

$$V_i(c_i, x_i) = \int_0^\infty e^{-\rho t} \left[\frac{c_i(t)^{1-\alpha} - 1}{1-\alpha} - \frac{x_i(t)^{1+\gamma}}{(1+\gamma)\theta}\right] dt; \tag{4}$$

[11] Under the development or transition interpretation, the communal endowment may owe to a change in governance or regulation that weakens property rights. Under the environmental interpretation, technological progress may allow its extraction.

[12] Dockner, Jørgenson, Long, and Sorger (2000, p. 40, Definition 3.1) use 'feasible' in place of 'admissible'.

where $\theta > 0$ is an extraction cost parameter; $\alpha > 0$ ensures that consumption utility is increasing and concave; $\gamma > 0$ makes extraction costs increasing and convex.

The constraints facing agents maximising objective function 4 depend on their institutional environment. Without capital markets, agents are under intertemporal autarky and face instantaneous budget constraints: $c_i(t) \leq x_i(t) \, \forall t \in [0, \infty)$. They also face feasibility constraint 2.

Capital market access does not alter the feasibility constraint, but does replace the instantaneous budget constraints with a single intertemporal budget constraint:

$$\int_0^\infty e^{-rt} \left[c_i(t) - x_i(t) \right] dt \leq 0. \tag{5}$$

Thus, $x_i(t) < c_i(t)$ implies either borrowing against future extraction or dissaving, depending on the sign of $\int_0^t e^{-rt} \left[x_i(\tau) - c_i(\tau) \right] d\tau$. The opposite implies either repaying past consumption or saving for future consumption. Borrowing and lending take place at the same exogenous rate, r. (When $a < r$, the absence of property rights over the low yield resource is not so problematic.) The exogeneity of r makes the model a partial equilibrium model.[13] Default is not permitted.

Unlike a budget constraint in a Walrasian economy, the present constraint is not linear in the endogenously determined state variable, T in this case.

As instantaneous utility is unbounded, integral 4 may not converge as $t \to \infty$. To avoid comparisons of infinite valuations, we impose Uzawa integrability conditions that ensure finite valuations:

$$(1 - \alpha) r < \rho < (1 + \gamma) r. \tag{U}$$

These ensure that, respectively, the utility of consumption is finite, and that the disutility of extraction is finite when $T = \infty$.[14]

Finally, a *perfect foresight equilibrium* is a triple of time paths $(k, c_i, x_i) : t \to \Re^3_+$ such that

1. agents choose consumption and extraction paths to maximise utility subject to their budget and feasibility constraints;

2. the evolution of the commons stock is determined by the aggregate extraction path.

[13] Cai and Treisman (2005) endogenously determines interest rates in a static model.

[14] See Dockner et al. (2000, pp. 62 - 63) for a discussion of optimality criteria when valuations are infinite.

A perfect foresight equilibrium is a special case of a rational expectations equilibrium without private information or exogenous shocks. It is competitive in the standard sense: although agents' aggregate behaviour influences the economic environment, $k(t)$ and T, they disregard their individual effects on it. Thus, agents are extinction date takers.

3 Perfect foresight equilibria

Assuming that individuals can borrow and lend at rate r allows their problem of determining extraction and consumption to be broken into two separate problems:

1. the consumption-smoothing problem: choose $c_i(t)$ given total wealth $X_i(r) = \int_0^\infty e^{-rt} x_i(t) dt$ subject to constraint 5; and
2. the effort-smoothing problem: choose $x_i(t)$, given the extinction date T and feasibility constraint, 2.

3.1 The consumption smoothing problem

Agents are viewed as first solving the consumption problem:

$$\max_{c_i(t)} \int_0^\infty e^{-\rho t} \frac{c_i(t)^{1-\alpha} - 1}{1-\alpha} dt \text{ subject to equation 5.} \tag{6}$$

Defining

$$X_i(r) \equiv \int_0^\infty e^{-rt} x_i(t)\, dt;$$

facilitates writing the Lagrangian:

$$\mathcal{L} \equiv \int_0^\infty e^{-\rho t} \frac{c_i(t)^{1-\alpha} - 1}{1-\alpha} dt - \lambda \left[\int_0^\infty e^{-rt} c_i(t)\, dt - X_i(r) \right].$$

The ensuing Euler equation is standard:

$$c_i(t) = \frac{\rho - (1-\alpha) r}{\alpha} X_i(r) e^{\frac{r-\rho}{\alpha} t}. \tag{7}$$

This is independent of a, γ and θ. Its constant term is determined by the constraint. Substitution into the objective function of equation 6 therefore produces the maximized present value of utility from consumption

$$U_B(X_i(r)) \equiv \frac{1}{1-\alpha} \left[\nu X_i(r)^{1-\alpha} - \frac{1}{\rho} \right]; \tag{8}$$

where

$$\nu \equiv \left(\frac{\alpha}{\rho-(1-\alpha)r}\right)^{\alpha} = \frac{1}{(r-g_c)^{\alpha}}.$$

The Uzawa finite valuation condition for consumption is $\rho > r(1-\alpha)$. This is trivially satisfied if $\alpha \geq 1$, so that consumption utility is more concave than the logarithmic function. We note that

$$c_i(t) = c_i(0)e^{g_c t};$$

where

$$g_c \equiv \frac{r-\rho}{\alpha};$$

is the growth rate of consumption and $c_i(0)$ is chosen to satisfy (5):

$$c_i(0) = (r-g_c)\, X_i(r). \tag{9}$$

The Uzawa condition may therefore be expressed as $g_c < r$: consumption growth is less than the interest rate.

3.2 The effort smoothing problem

The agent's problem is now to

$$\max_{x_i(t)\geq 0} U_B\left(\int_0^{\infty} e^{-rt}x_i(t)\,dt\right) - \frac{1}{\theta}\int_0^{\infty} e^{-\rho t}\frac{x_i(t)^{1+\gamma}}{1+\gamma}dt; \tag{10}$$

subject to feasibility constraint 2.

When $t < T$, the ensuing Euler equation yields

$$x_i(t) = e^{gt}\frac{(\nu\theta)^{\frac{1}{\gamma}}}{X_i(r)^{\frac{\alpha}{\gamma}}} = e^{gt}\frac{(\nu\theta)^{\frac{1}{\gamma}}}{\left[\int_0^T e^{-rt}x_i(t)\,dt\right]^{\frac{\alpha}{\gamma}}} = e^{gt}\kappa;$$

where $g = \frac{\rho-r}{\gamma}$ and κ is a positive constant. Evaluating this at $t=0$ produces $x_i(0) = \kappa$ so that

$$x_i(t) = \left\{\begin{array}{lll} x_i(0)e^{gt} & \text{for} & t<T \\ 0 & \text{for} & t\geq T \end{array}\right\}. \tag{11}$$

This allows the term in $x_i(t)$ to be removed from the integral for

$$x_i(0)^{\alpha+\gamma} = \frac{\nu\theta}{Q_{r-g}(T)^{\alpha}}; \tag{12}$$

where

$$Q_n(T) \equiv \int_0^T e^{-n\tau} d\tau = \frac{1 - e^{-nT}}{n}$$

for $n \neq 0$ and $T \geq 0$.

Thus, extraction is smooth until the extinction date, T. More significantly, the problem of choosing an extraction path is reduced to a choice of $x_i(0)$. Note also that the extraction plan is a function of α: thus, full Fisher separation of extraction (production) and consumptions plans does not occur. This is a consequence of $\gamma > 0$: extraction costs are borne as non-transferable disutility.[15]

The Uzawa condition for extraction may therefore be expressed as $g < r$.

Notice that $g = -\frac{\alpha}{\gamma} g_c$; thus, for $r > \rho$, individuals extract early but consume late. When $r = \rho$, so that the interest rate equals the subjective discount rate, $g = g_c = 0$.

Notice also that an expression for extraction as a function of the communal endowment may now be written. By equations 3 and 11,

$$k(t) = e^{at}\left[k(0) - \frac{x(0)}{g-a}\left(e^{(g-a)t} - 1\right)\right].$$

Thus, when $g \neq a$, extraction cannot be expressed as a linear function of the endowment. Thus, the modelling in Tornell and Velasco (1992) and Tornell and Lane (1999) is restrictive.

3.3 Characterising equilibrium

Rewrite equation 12 as

$$x_i(0) = \frac{\mu}{\left(Q_{r-g}(T)\right)^{\frac{\alpha}{\alpha+\gamma}}}; \tag{13}$$

where $\mu \equiv (\nu\theta)^{\frac{1}{\alpha+\gamma}}$.

Integrating over agents then produces the first fundamental equation: the effect of anticipated extinction on extraction:

$$x(0) = \frac{\mu}{\left(Q_{r-g}(T)\right)^{\frac{\alpha}{\alpha+\gamma}}}. \tag{A}$$

This equation gives us a monotone decreasing map $T \to x(0)$.

[15] In the limit, as $\gamma \to 0, x_i(0)$ ceases to depend on α by exploding to infinity. This will also be demonstrated in Theorem 3.

We now obtain the second fundamental equation: the impact of extraction on the extinction date of the commons. As extinction occurs at the lowest T such that $k(T) = 0$, it represents a zero of equation 3:

$$k(0)\, e^{aT} = \int_0^T e^{a(T-\tau)} x(\tau)\, d\tau.$$

By equation 11 and integration over agents,

$$k(0) = x(0) \int_0^T e^{-(a-g)\tau} d\tau. \tag{14}$$

To simplify exposition, we assume here that $a > r$.[16] With the Uzawa extraction condition, $r > g$, it follows that $a > g$. Therefore

$$\frac{k(0)}{x(0)} = \frac{1 - e^{-(a-g)T}}{a-g} = Q_{a-g}(T) \leq \frac{1}{a-g}.$$

This expression reaches its upper bound as $T \to \infty$. Therefore T implicitly solves

$$Q_{a-g}(T) = \min\left\{\frac{k(0)}{x(0)}, \frac{1}{a-g}\right\}. \tag{I}$$

This equation gives us a map $x(0) \to T$, also monotone non-increasing. Here a low $x(0)$ guarantees the perpetuation of common property but a level higher than $(a-g)k(0)$ results in extinction in finite time.

Both maps are decreasing: individuals choose to extract more if they believe that the commons will disappear soon; and higher extraction rates speed up extinction. Such strategic complementarities are standard sources of multiple equilibria (Vives, 2005).

To formalise the preceding discussion, define

$$\Psi(t) \equiv \frac{Q_{a-g}(t)}{[Q_{r-g}(t)]^{\frac{\alpha}{\alpha+\gamma}}}; \tag{15}$$

and $\psi_* \equiv \lim_{t\to\infty} \Psi(t), \psi^* \equiv \max_t \Psi(t)$.

Theorem 1. *There exist $0 < k_L \leq k_H \leq \infty$ such that the following statements hold when agents have access to capital markets. Given intervals $I_L = [0, k_L), I_M = (k_L, k_H)$, and $I_H = (k_H, \infty)$:*

1. *$k(0) \in I_L$ implies unique equilibrium with finite extinction;*

[16] Theorem 2 explicitly considers the complementary case.

2. *$k(0) \in I_M$ implies three equilibria, one with non-extinction and two with finite extinction;*

3. *$k(0) \in I_H$ implies unique equilibrium without extinction;*

4. *$k(0) = k_L = k_H$ implies a unique equilibrium with non-extinction;*

5. *$k(0) \in \{k_L, k_H\}$ for $k_L \neq k_H$, implies a unique equilibrium with extinction, and a unique equilibrium with non-extinction.*

Consistent with the usual fixed point arguments, we therefore generically find an odd number of equilibria.

The following lemmas are used to prove the Theorem:

Lemma 1. $\lim_{t\to 0} \Psi(t) = 0$.

Proof. As $Q_n(0) = 0$, assessing $\Psi(0)$ requires use of l'Hôpital's rule: differentiating the numerator produces 1, while doing so to the denominator produces

$$\frac{\alpha}{\alpha+\gamma}\left[Q_{r-g}(0)\right]^{-\frac{\gamma}{\alpha+\gamma}} e^{-(r-g)0} = \frac{1}{0} = \infty.$$

□

Lemma 2. *When the Uzawa extraction condition holds and $a > g, 0 < \psi_* < \infty$.*

Proof. By definition, $\lim_{t\to\infty} Q_n(t) = \frac{1}{n}$ when $n > 0$.[17] Under the conditions of the lemma,

$$\psi_* = \frac{(r-g)^{\frac{\alpha}{\alpha+\gamma}}}{a-g}. \tag{16}$$

The Uzawa extraction condition ensures that the numerator is strictly positive. When $a > g$, the denominator is as well, ensuring the results. □

Lemma 3. *An equilibrium with finite extinction time T satisfies $\Psi(T) = \frac{k(0)}{\mu}$.*

Proof. Equations A and I, with $Q_{a-g}(T) = \frac{k(0)}{x(0)}$, are satisfied in equilibrium. The result follows by definition 15. □

Lemma 4. *An equilibrium with non-extinction exists iff*

$$\psi_* \leq \frac{k(0)}{\mu}.$$

[17]When $n \leq 0$ the limit is infinite.

Proof. Assume that $\psi_* \leq \frac{k(0)}{\mu}$ corresponds to a $T = \infty$ equilibrium. Therefore, by definition,

$$\begin{aligned} \frac{k(0)}{\mu} &\geq \frac{Q_{a-g}(\infty)}{[Q_{r-g}(\infty)]^{\frac{\alpha}{\alpha+\gamma}}} \\ &= Q_{a-g}(\infty)\frac{x(0)}{\mu} \text{ by equation A;} \end{aligned}$$

so that rearrangement produces

$$\frac{k(0)}{x(0)} \geq Q_{a-g}(\infty) = \frac{1}{a-g};$$

which satisfies equation I. Thus, the conditions for equilibrium are satisfied. By contrast, if $\psi_* > \frac{k(0)}{\mu}$, the final inequality above does not satisfy equation I. □

Defining $k_L \equiv \mu\psi_*$ and $k_H \equiv \mu\psi^*$ allows the Theorem to be proved. This is done in the Appendix. Figure 1 illustrates extinction dates. (Multiply the horizontal axis by μ to replace $\Psi(t)$ by $k(0)$.)

While Lemma 2 assumed $a > g$, that condition is sufficient but not necessary for the proof of the Theorem. Thus, the Theorem holds more generally. While $a > r$ presents the most interesting class of cases, its complement contains some canonical cases. Non-renewable resources, for which $a = 0$, is the most obvious. As these cases may be analysed using the objects already developed, the following theorem is easily proved:

Theorem 2. *When the Uzawa extraction condition holds:*

1. I_L *is always non-empty.*
2. I_M *is non-empty iff* $a > r$.
3. I_H *is empty when* $a \leq g$.

The proof makes use of the following lemma:

Lemma 5. *For* $\Psi(t)$ *to be strictly quasiconcave, either of the following are sufficient:*

1. $a \neq g$;
2. $\alpha > 0$ *and the Uzawa extraction condition holds.*

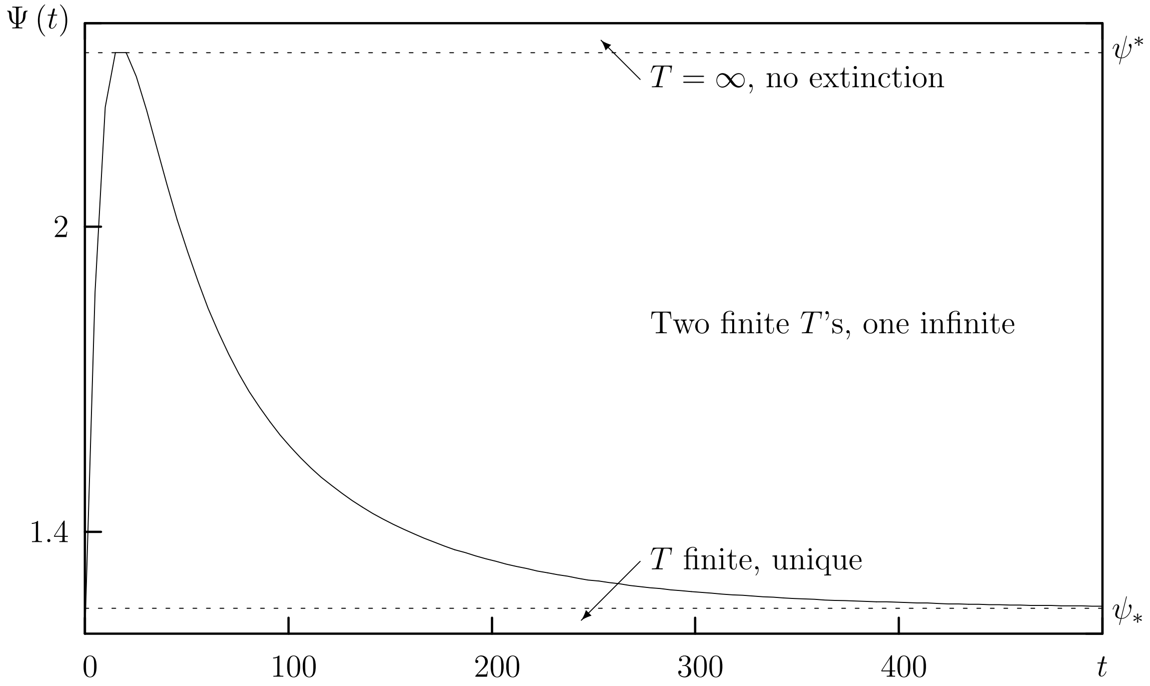

Figure 1: Extinction dates when $\alpha = \gamma = 1, \rho = \frac{1}{20}, a = \frac{1}{10}, r = \frac{3}{100}$

Proof. Define

$$D(t) \equiv \frac{d \ln \Psi(t)}{dt} = \frac{a-g}{e^{(a-g)t}-1} - \frac{\alpha}{\alpha+\gamma} \frac{r-g}{e^{(r-g)t}-1}. \quad (17)$$

Therefore

$$D'(t) = -\frac{(a-g)^2}{\left[e^{(a-g)t}-1\right]^2} - \frac{\alpha}{\alpha+\gamma} \frac{(r-g)^2}{\left[e^{(r-g)t}-1\right]^2}.$$

By l'Hôpital's rule, when $a = g$ or $r = g$, the whole term in which it is contained is zero. Thus, the stated conditions of the lemma suffice to ensure that either the first or second term of $D'(t)$ is negative.

If either term is non-zero, the whole expression is strictly negative. This suffices for $\ln \Psi(t)$ to be strictly concave and, thus, for $\Psi(t)$ to be strictly quasiconcave. □

The Theorem's second result identifies the source of multiple equilibria. When $a \leq r$, there is no incentive to conserve the commons: the complementarity that leads agents to slow their extraction when $a > r$ in response to an increased extinction date no longer exists. The third result is also intuitive:

if the growth rate of extraction exceeds that of the commons, the commons will eventually be extinguished.

In closing, consider the limit case of $\gamma \to 0$, linear extraction costs, so that extraction costs reflect only total extraction, rather than the rate of extraction. This may be interpreted as a situation in which there is a competitive market for the inputs into a CRS extraction function.

Theorem 3. *For generic endowments, $k(0)$, the extinction date, T, goes to zero with γ.*

Intuitively, removing convexity from the extraction costs removes a disincentive to pace one's own extraction, inducing a rush to deplete the commons. The result parallels that in Gaudet et al. (2002), in which, once reserves fall to a critical level, "the extraction contest is so fierce that the common is drained in the instant storage is initiated" when average costs are constant. Here, $\gamma \to 0$ makes both marginal and average costs constant. Unlike Gaudet et al. (2002), extinction is instantaneous here, rather than occurring after a period of slower extraction. The difference between these results does not reflect the difference between storage alone and full capital market access: when $\gamma \to 0$ our agents also store their income; they do not borrow against future income.

4 The commons without capital markets

This section compares equilibria with capital market access to those without such access. This latter environment is the one usually analysed in the dynamic commons literature.

Without access to capital markets, individuals cannot disentangle their consumption and effort smoothing problems: a single instrument must be used to solve both smoothing problems. Intertemporal budget constraint 5 is replaced by $c_i(t) = x_i(t)$; feasibility constraint 2 remains the same. Thus, agents maximise

$$\tilde{V}_i(\tilde{x}_i) = \int_0^\infty e^{-\rho t}\left[\frac{\tilde{x}_i(t)^{1-\alpha}-1}{1-\alpha} - \frac{\tilde{x}_i(t)^{1+\gamma}}{(1+\gamma)\theta}\right] dt \text{ subject to constraint 2;}$$

where tildes distinguish autarkic variables from those with capital market access. Maximising produces

$$\tilde{x}_i(t) = \left\{ \begin{array}{lll} \theta^{\frac{1}{\alpha+\gamma}} & \text{for} & t < \tilde{T} \\ 0 & \text{for} & t \geq \tilde{T} \end{array} \right\}; \tag{18}$$

The constant extraction rate implies finite valuation without further conditions.

Integrating over all agents yields aggregate extraction

$$\tilde{x}(t) = \left\{ \begin{array}{lll} \theta^{\frac{1}{\alpha+\gamma}} & \text{for} & t < \tilde{T} \\ 0 & \text{for} & t \geq \tilde{T} \end{array} \right\}. \tag{19}$$

Agents therefore treat the optimisation problem as a static one: the inability to intertemporally smooth and the absence of strategic interaction eliminates dynamic aspects from their problem. This, in turn, eliminates the possibility of strategic complementarities. Extinction dates, in contrast to those derived in Theorem 1 under capital market access, are unique:

Theorem 4. *When agents do not have access to capital markets, there is a unique, finite extinction date iff*

$$k(0) < \frac{1}{a}\theta^{\frac{1}{\alpha+\gamma}}.$$

Otherwise, the unique equilibrium has no extinction.

The proof, in the Appendix, shows that $\tilde{T}$ is convex in $k(0)$. Again, the Uzawa conditions derived earlier are assumed to hold.

Figure 2 displays an example of the effect of capital market access on extinction dates.[18] The curve referring to capital market access is that in Figure 1. Here, low levels of commons stock are preserved for longer by individuals under autarky. Above $k(0) = \mu\psi_*$, an intermediate zone is entered. In this, the autarkic extinction date lies between the two finite extinction dates with capital market access. At the same time, there is a market access equilibrium with no extinction.

Finally, above a higher level of $k(0)$, the autarkic extinction date is greater than both of the finite dates with access. Again, though, there is a non-extinction equilibrium with capital market access.

Comparing extinction dates does not provide much insight into welfare comparisons: extinction under autarky ensures no further consumption as well as extraction; under market access, it merely halts extraction. We therefore compare welfare directly.

The equilibrium welfare obtained by the infinitesimal agents di under market access may be expressed in terms of initial extraction, $x_i(0)$ by substitution of equations 8, 11 and 12 into equation 4:

$$W_i(k(0)) = \frac{1}{1-\alpha}\left\{\frac{\alpha+\gamma}{1+\gamma}\nu^{\frac{1+\gamma}{\alpha+\gamma}}\theta^{\frac{1-\alpha}{\alpha+\gamma}}\left[Q_{r-g}(T)\right]^{\frac{1-\alpha}{\alpha+\gamma}\gamma} - \frac{1}{\rho}\right\}. \tag{20}$$

[18] Maple code available at www.economics.bham.ac.uk/rowat/research/060224allcalcs.mw.

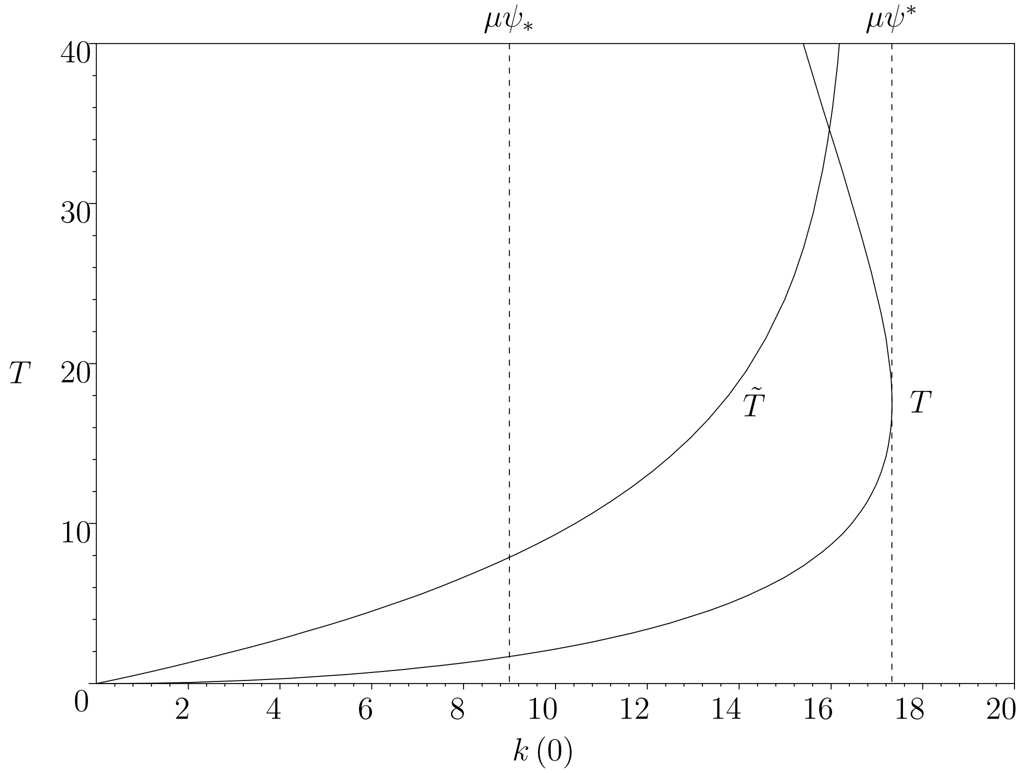

Figure 2: Extinction dates varying in $k(0)$ when $\alpha = \gamma = 1, \rho = \frac{1}{20}, \theta = e, a = \frac{1}{10}, r = \frac{3}{100}$

As T is not generally a function of $k(0)$, this cannot be expressed as a function of $k(0)$ directly.

The corresponding autarkic welfare obtained by individuals di is:

$$\tilde{W}_i(k(0)) = \frac{1}{(1-\alpha)\rho}\left\{\left[\frac{\alpha+\gamma}{1+\gamma}\theta^{\frac{1-\alpha}{\alpha+\gamma}} - 1\right]\left[1 - e^{-\rho\tilde{T}}\right] + e^{-\rho\tilde{T}}\left[0^{1-\alpha} - 1\right]\right\}. \tag{21}$$

The following theorem compares welfare under market access and autarky. The cases that it presents avoid the problem of multiple equilibria found in Theorem 1.

Theorem 5. *When*

1. $k(0) \geq \max\left\{\frac{1}{a}\theta^{\frac{1}{\alpha+\gamma}}, \mu\psi^*\right\}$*, market access dominates autarky whenever* $\rho \neq r$*. When* $\rho = r$*, welfare is identical.*

2. $k(0) < \frac{1}{a}\theta^{\frac{1}{\alpha+\gamma}}$ *and* $\alpha \geq 1$*, market access dominates autarky.*

3. $k(0) \geq \frac{1}{a}\theta^{\frac{1}{\alpha+\gamma}}$*, autarky dominates market access at* $\gamma \to 0$ *iff* $\rho \geq r$*.*

4. *$k(0) < \frac{1}{a}\theta^{\frac{1}{\alpha+\gamma}}$ and $\alpha < 1$, a necessary condition for autarky to dominate market access at $\gamma \to 0$ is $\rho > r$. Autarky then dominates iff $k(0)$ lies above a critical value.*

5. *$k(0) < \min\left\{\frac{1}{a}\theta^{\frac{1}{\alpha+\gamma}}, \mu\psi^*\right\}$, a sufficient condition for autarky to dominate market access at $\alpha \to 0$ is that $a > r$, ρ not be too much larger than r and $k(0)$ not be too much smaller than $\frac{1}{a}\theta^{\frac{1}{\alpha+\gamma}}$.*

6. *$\alpha < 1$, market access dominates autarky for small $k(0)$, reaching indifference at $k(0) = 0$.*

For the first result, as agents interact exclusively through the finite extinction date, setting it to infinity removes their interaction. Market access is generally preferred under these circumstances as it enlarges agents' action space without strategic consequence. When $\rho = r$, extraction and consumption under market access are not only constant, but equal to that under autarky. Thus, when the market and subjective discount rates are the same, intertemporal markets are not used - a *pseudo-autarkic equilibrium.*

Intuition for the second result is immediate: a finite extinction date and consumption utility more curved than the logarithmic delivers payoffs of negative infinity beyond extinction.

To understand the third result, recall Theorem 3: $\gamma \to 0 \Rightarrow T \to 0$, so that no use is made of the high return commons under market access. Perhaps surprisingly, market access still outperforms autarky when agents are patient. Although $\gamma \to 0$ initially forces a frenzy of extraction, their patience allows these initial costs to be outweighed by subsequent benefits of market access. This is not so when they are impatient. Equivalently, in terms of the interest rate: when r is low, agents earn little on the resources that they initially extract, leading them to prefer autarky.

The intuition for the fourth result is the same. The condition on α prevents negative infinite payoffs under autarky.

Regarding the fifth result, at $\alpha \to 0$ and $\rho > r$ ($\rho < r$ violates the Uzawa consumption condition, deferring consumption until $t = \infty$), all consumption under market access occurs instantly. As to extraction, $x_i(0) = \tilde{x}_i(0)$, but the former grows exponentially at rate g for all $t \leq T$ thereafter, while the latter remains constant. When $\rho = r$, then, the extraction paths are identical. At this limit, the consumption plans under autarky and market access have the same value: linear utility and equal subjective and market discount rates mean that the scheduling of consumption is irrelevant. Here, then, agents are indifferent between market access and autarky. This is a weaker form of pseudo-autarky as consumption paths may differ under the two institutions, but irrelevantly.

An increase in ρ has two effects on the market access equilibria. First, and positively, it discounts the (future) extraction costs while leaving the (present) consumption utility unaltered. Second, negatively when $a > r$, it speeds the commons' extinction. As $\rho \rightarrow (1+\gamma) r$ the former effect dominates, so that market access, in turn, dominates autarky. For large $k(0)$, the autarkic $\tilde{T}$ is very large, as illustrated in Figure 2. When ρ just exceeds r, the latter effect dominates: slight impatience substantially speeds extinction.

The relative performance of market access and autarky therefore depends not just on the magnitude of ρ relative to r but on α and γ. When $\gamma \rightarrow 0$, agents are creditors under market access, benefitting from high r. When $\alpha \rightarrow 0$, agents with market access are debtors, benefitting from low r.

The more general intuition to draw is that removing one of the agents' smoothing problems - by setting α or γ to zero - allows autarky (in which agents have only a single instrument) to outperform market access (in which agents determine extraction and consumption plans) under certain conditions. Those conditions depend on whether agents would be net creditors or debtors under market access. When creditors ($\gamma \rightarrow 0$), high impatience relative to the interest rate benefits autarky. When debtors ($\alpha \rightarrow 0$), high impatience relative to interest rates would discount future repayments, weighing against autarky.

Results for low values of α and γ are of greater practical interest, but involve less tractable calculations. In the extreme case, when α and γ both go to zero, market access and autarky yield identical welfare, $-\frac{1}{\rho}$.

Finally, when zero consumption still delivers finite utility, market access dominates autarky when the communal endowment is sufficiently scarce. Now both technologies quickly deplete the commons, but the former allows the proceeds to be consumed when desired.

Figure 3 illustrates the Theorem's first and last results. It perturbs the parameter values used above very slightly, setting $\alpha = \frac{99}{100}$. This ensures that welfare under autarky does not drop to $-\infty$ with finite extinction.[19] The Figure also identifies a region in which autarky may dominate market access that the Theorem did not: when $\tilde{T}$ is infinite, but market access yields multiple equilibria autarky is preferred to the market access outcomes with finite extinction. The mechanism used to select among market access' multiple equilibria will therefore bias in favour of or against market access.[20]

[19]When $\alpha = 1$, the graph is largely unchanged, but $\tilde{W}_i(k(0))$ rises discontinuously from $-\infty$ at $k(0) \approx 16.5$, where $\tilde{T}$ becomes infinite.

[20]While reasoning from global games suggest that the risk dominant market access equilibria might be selected (Morris and Shin, 2003), the argument is merely suggestive at present: if, for example, agent i 'played' the $x_i(0)$ associated with $T = \infty$ when the actual T generated by agents $-i$ was finite, agent i might violate its intertemporal budget

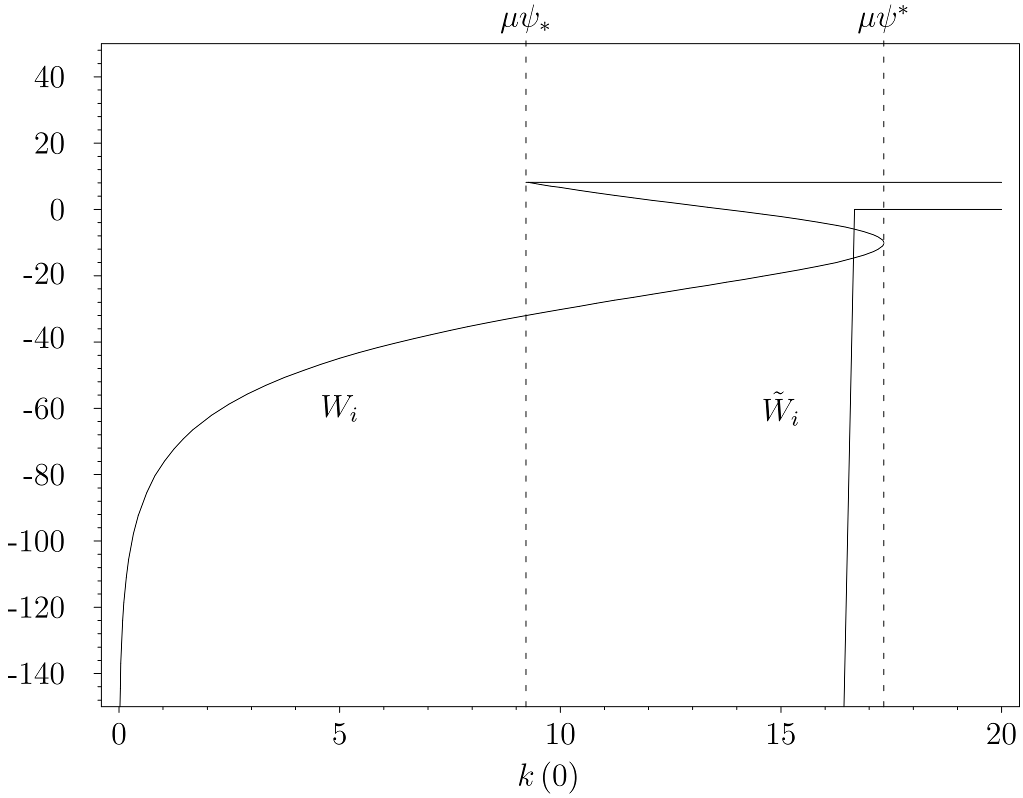

Figure 3: Welfare varying in $k(0)$ when $\alpha = \frac{99}{100}, \gamma = 1, \rho = \frac{1}{20}, \theta = e, a = \frac{1}{10}, r = \frac{3}{100}$

The final point illustrated by Figure 3 also relates to welfare under market access. In the region of multiple equilibria, welfare may increase, decrease or remain constant as the communal endowment increases. Along the highest branch, $T = \infty$: welfare is insensitive to changes to $k(0)$ as agents do not regard themselves as constrained. Along the lowest branch, welfare increases with $k(0)$, an intuitive result. Along the intermediate branch, however, it decreases. We identify this as a form of 'resource curse' (q.v. Sala-i-Martin and Subramanian, 2003; Bannon and Collier, 2003): higher levels of endowment support equilibria in which agents correctly expect that they will deplete the commons more energetically, extinguishing it more quickly. Nevertheless, the cursed branch still yields higher welfare than does the upward sloping branch.

constraint.

5 Discussion

We have analysed extraction from a commons when agents have access to capital markets. Comparison to the standard in the literature, in which agents do not have such access, shows that the results can differ significantly: multiple equilibria may arise, against unique extinction dates under autarky; thus, welfare generically differs in the two environments.

When communal resources are sufficiently abundant that they would not be depleted under autarky, the market is preferred for the intertemporal transfers that it allows. Otherwise, the model identifies a number of 'risk factors' that may lead to autarky's superiority. First, autarky performs well with constant returns to extraction ($\gamma \to 0$), and impatient agents. Second, the absence of consumption risk aversion ($\alpha \to 0$) can also favour autarky.

If CRS extraction and high subjective discount factors are typical of polities with weak property rights, these conclusions are cautionary. For small agents, burning fossil fuels to emit into the atmosphere, looting government facilities or transmission towers for scrap, may seem like CRS activities. On the other hand, if resource scarcity is their more important characteristic, market access may offer substantial improvements over autarky.

Adjudication between these two possibilities requires empirical analysis, an obvious extension of this work. A good measure of communal endowment, $k(0)$, should involve both physical and natural capital. Data on the latter may be especially poor, however, in countries with weak property rights.

A second extension to consider are mechanisms for hardening property rights. As these involve assigning capital stocks to each agent (with, presumably, a lower extraction cost from one's own stock), they imply a larger state space. Even in the commons, analysis of strategic agents generally requires that the state variable include some form of both private savings and communal endowment.

A policy initiative that can be easily considered in the current framework is an extraction tax, δ, so that infinitesimal agents di retain $(1-\delta)\, x_i^\delta$ of their extracted x_i^δ. Following the interpretation in Tornell and Lane (1999), tax revenues raised by the state are then contested by parties competing for the state's resources: thus, tax revenue, δx^δ, is returned to the commons, so that taxes both reduce agents' productivity and replenish the commons. Rewriting the analysis of Section 3 shows that the tax 'inflates' $k(0)$, delaying extinction. This and the direct effect of the tax both work to reduce initial extraction, $x^\delta(0)$. Rewriting equation 20 reveals two effects on welfare:

$$W_i^\delta = \frac{1}{1-\alpha}\left[\frac{\alpha+\gamma}{1+\gamma}\nu^{\frac{1+\gamma}{\alpha+\gamma}}\theta^{\frac{1-\alpha}{\alpha+\gamma}}(1-\delta)^{\frac{(1-\alpha)(1+\gamma)}{\alpha+\gamma}} Q_{r-g}\left(T^\delta\right)^{\frac{1-\alpha}{\alpha+\gamma}\gamma} - \frac{1}{\rho}\right]. \quad (22)$$

The direct effect of a consumption tax reduces welfare, while the indirect effect through the extinction date increases it. In a market access equilibrium without extinction, a consumption tax therefore does not improve welfare.[21]

We conclude by mentioning three further possible extensions. First, generalising the return to $a(k)$ would be more consistent with both biological interpretations of the model (q.v. Dockner and Sorger, 1996; Kremer and Morcom, 2000) and economies of scale. (This could also capture some of the additional costs of extracting resources as they become scarce.) This generalisation leaves equation A unchanged but complicates equation I.

Second, allowing the communal endowment to enter directly in instantaneous utility (as in Long and Sorger (2006)) also seems more consistent with biological interpretations of the model: $k(t)$ might provide eco-system services directly.

Finally, the absence of default provisions is an obvious limitation of the present analysis.

A Appendix

Proof of Theorem 1. When $k(0) \in I_L$, $k(0) \geq 0$ and $\psi_* > \frac{k(0)}{\mu}$. Lemmas 1 and 2 and the continuity of $\Psi(t)$ ensure that there is a single finite T such that $\Psi(T) = \frac{k(0)}{\mu}$. By Lemma 3, this implies a unique equilibrium with finite extinction date. As the inequality in ψ_* is the reverse of the necessary and sufficient condition in Lemma 4, there are no equilibria with infinite extinction dates.

Now consider $k(0) \in I_M \Rightarrow \psi_* < \frac{k(0)}{\mu} < \psi^*$. The first of these ensures, by Lemma 4, the existence of an equilibrium with an infinite extinction date. For the second inequality to hold, it must be that $\psi_* < \psi^*$. By the continuity of $\Psi(t)$ and the definition of ψ^*, there are two finite T such that $\Psi(T) = \frac{k(0)}{\mu} < \psi^*$. By Lemma 3, these are equilibria with finite extinction times.

When $k(0) \in I_H$, $\frac{k(0)}{(\nu\theta)^{\frac{1}{\alpha+\gamma}}} > \psi^*$. Thus, by Lemma 3, there are no equilibria with finite extinction times; by Lemma 4, there is one without extinction.

Now consider the degenerate cases. First, $k(0) = k_L = k_H \rightarrow \frac{k(0)}{\mu} = \psi_* = \psi^*$. By Lemma 3, there is no equilibrium with finite extinction as ψ_*

[21] A consumption tax, reducing consumption to $(1-\varepsilon) c_i^{\varepsilon}$, can also considered reasonably easily: If the proceeds are returned to the commons equation of motion 1 becomes more complicated, complicating, in turn, the new version of equation I. Savings taxes or instruments resembling capital controls are more difficult to consider. By taxing either the absolute value of $x_i(t) - c_i(t)$ or its positive component, a kink is introduced into the consumption smoothing problem.

is only reached as $T \to \infty$. Lemma 4 is satisfied with equality, producing an equilibrium without extinction.

Finally, when $k_L \neq k_H$, Lemma 4 is satisfied. Now a single finite T satisfies Lemma 3, tangentially when $k(0) = k_H$. □

Proof of Theorem 2.

1. the continuity of $\Psi(t)$ and $\Psi(0) = 0$ ensure the result if $\psi_* > 0$. When $a > g$, this has already been demonstrated in Lemma 2. When $a = g$ and $r > g$,

$$\Psi(t) = t\left[\frac{r-g}{1-e^{-(r-g)t}}\right]^{\frac{\alpha}{\alpha+\gamma}}. \tag{23}$$

Thus, the Uzawa extraction condition ensures that $\psi_* = \infty$. Finally, when $a < g, \lim_{t\to\infty} Q_{a-g}(t) = \infty$; as $\lim_{t\to\infty} Q_{r-g}(t), \psi_*$ is again infinite.

2. Sufficient conditions for the existence of I_M are that $\Psi'(t) = 0$ for a finite t and that $\Psi(t)$ be strictly quasiconcave. Consider all possible cases.

Under the Uzawa extraction condition, $a = g$ sets

$$\Psi'(t) = \frac{1}{[Q_{r-g}(t)]^{\frac{\alpha}{\alpha+\gamma}}}\left[1 + \frac{\alpha}{\alpha+\gamma}\frac{te^{-(r-g)t}}{Q_{r-g}(t)}\right] > 0 \forall t > 0.$$

Thus, $r > a = g$ suffices for an empty I_M.

Now consider $a \neq g$. Here

$$\Psi'(t) = \frac{e^{gt}}{Q_{r-g}(t)^{\frac{\alpha}{\alpha+\gamma}}}\left[e^{-at} - \frac{\alpha}{\alpha+\gamma}\frac{Q_{a-g}(t)}{Q_{r-g}(t)}e^{-rt}\right]. \tag{24}$$

Thus, a stationary point sets the square bracketed term to zero. Equivalently, it solves

$$\frac{e^{rt}-e^{gt}}{e^{at}-e^{gt}} = \frac{\alpha}{\alpha+\gamma}\frac{r-g}{a-g}. \tag{25}$$

To simplify analysis, define

$$\xi(t) \equiv \frac{e^{rt}-e^{gt}}{e^{at}-e^{gt}}.$$

Thus, $\xi(t)$ is continuous for all $t \geq 0$ and, by l'Hôpital's rule, $\xi(0) = \frac{r-g}{a-g}$. As this is greater in absolute value than the right hand side of equation 25 for all $\gamma > 0$, a sufficient condition for an empty I_M is that $(a-g)\xi'(t) \geq 0 \forall t$.

Calculation yields

$$\xi'(t) = \frac{(r-a)\,e^{(a+r)t} - (r-g)\,e^{(r+g)t} + (a-g)\,e^{(a+g)t}}{\left(e^{at} - e^{gt}\right)^2}. \tag{26}$$

When $r > a > g$, this is positive.

Now consider $r > g > a$. By Lemma 5, $\Psi(t)$ is strictly quasiconcave. As $t \to \infty$, its denominator tends to $\left(\frac{1}{r-g}\right)^{\frac{\alpha}{\alpha+\gamma}}$, a positive finite number. Its numerator, however, tends to infinity. This, by strict quasiconcavity, precludes a maximum in finite t. Thus, I_M is empty under these conditions.

Now consider $a = r > g$. In this case, the square bracketed term in equation 26 is identically zero, so that $\xi'(t) = 0 \forall t$. This suffices, from above, for an empty I_M.

Finally, consider $a > r > g$, the case considered above. In this case, the denominator of $\xi(t)$ grows more quickly than the numerator, so that $\xi(t)$ asymptotes to zero as $t \to \infty$.

3. from the first steps in the proof, $\psi_* = \infty$ when $a \leq g$. With the Uzawa extraction condition, this suffices for an infinite ψ^*.

□

Proof of Theorem 3. Assume that T is finite. Then, under the stated conditions, equations A and I combine to yield

$$\Psi(T) = \frac{k(0)}{\mu} = \frac{Q_{a-g}(T)}{Q_{r-g}(T)^{\frac{\alpha}{\alpha+\gamma}}}.$$

This may be rearranged and rewritten in terms of primitives for

$$\frac{k(0)}{\mu}\frac{r + a\gamma - \rho}{\left[(1+\gamma)r - \rho\right]^{\frac{\alpha}{\alpha+\gamma}}}\gamma^{-\frac{\gamma}{\alpha+\gamma}} = \left[1 - e^{-\frac{r+a\gamma-\rho}{\gamma}T}\right]\left[1 - e^{-\frac{(1+\gamma)r-\rho}{\gamma}T}\right]^{-\frac{\alpha}{\alpha+\gamma}};$$

so that, as $\lim_{\gamma\to 0}\gamma^{-\frac{\gamma}{\alpha+\gamma}} = 1$,

$$\frac{k(0)}{\mu} = \lim_{\gamma\to 0}\left[1 - e^{-\frac{r+a\gamma-\rho}{\gamma}T}\right]\left[1 - e^{-\frac{(1+\gamma)r-\rho}{\gamma}T}\right]^{-\frac{\alpha}{\alpha+\gamma}}.$$

If T remained positive as $\gamma \to 0$ the right hand side of the equation would converge to unity, generically a contradiction. Thus, finite $T \to 0$ as γ does.

Now assume that T is infinite. Equation A requires $x(0) = \mu(r-g)^{\frac{\alpha}{\alpha+\gamma}}$. By the Uzawa extraction condition, $r - g > 0$; as $\gamma \to 0$, this explodes to infinity, a contradiction for finite $k(0)$. □

Proof of Theorem 4. Substitution of equation 19 into equation of motion 3 yields

$$e^{-a\tilde{T}} = 1 - \frac{ak\left(0\right)}{\theta^{\frac{1}{\alpha+\gamma}}}$$

when $k\left(t\right)$ is set to zero. Under the conditions of the theorem, this has the unique, finite solution

$$\tilde{T} = \frac{1}{a}\ln\left[\frac{\theta^{\frac{1}{\alpha+\gamma}}}{\theta^{\frac{1}{\alpha+\gamma}} - ak\left(0\right)}\right]. \tag{27}$$

Thus, as $k\left(0\right) \to \frac{1}{a}\theta^{\frac{1}{\alpha+\gamma}}$, $\tilde{T}$ approaches infinity asymptotically. □

Proof of Theorem 5. We prove the results in the order in which they are stated.

1. the condition ensures that $T = \tilde{T} = \infty$. By equations 20 and 21, $W_i\left(k\left(0\right)\right) \geq \tilde{W}_i\left(k\left(0\right)\right)$ is equivalent to

$$\frac{\rho}{1-\alpha}\left\{\frac{\alpha^{(1+\gamma)\alpha}}{\gamma^{(\alpha-1)\gamma}}\frac{\left[\left(1+\gamma\right)r-\rho\right]^{(\alpha-1)\gamma}}{\left[\rho-\left(1-\alpha\right)r\right]^{(1+\gamma)\alpha}}\right\}^{\frac{1}{\alpha+\gamma}} \geq \frac{1}{1-\alpha}. \tag{28}$$

When $\alpha < 1$, inequality 28 reduces to

$$\left[1-\frac{\left(1-\alpha\right)r}{\rho}\right]^{(1+\gamma)\alpha}\left[\frac{\left(1+\gamma\right)r}{\rho}-1\right]^{(1-\alpha)\gamma} \leq \alpha^{(1+\gamma)\alpha}\gamma^{(1-\alpha)}. \tag{29}$$

As ρ is isolated on its left hand side, a necessary condition for the inequality to hold is that maximising the LHS with respect to ρ does not cause it to exceed the RHS. The ensuing first order condition is

$$\frac{\left(1+\gamma\right)\alpha}{\rho-\left(1-\alpha\right)r} - \frac{\left(1-\alpha\right)\gamma}{\left(1+\gamma\right)r-\rho} - \frac{\alpha+\gamma}{\rho}.$$

Equating this to zero for an interior stationary point yields $\rho = r$, which satisfies the Uzawa conditions. The second order condition is

$$-\frac{\left(1+\gamma\right)\alpha}{\left[\rho-\left(1-\alpha\right)r\right]^2} - \frac{\left(1-\alpha\right)\gamma}{\left[\left(1+\gamma\right)r-\rho\right]^2} + \frac{\alpha+\gamma}{\rho^2};$$

which, at $\rho = r$, reduces to $\frac{\alpha+\gamma}{\alpha\gamma r^2}\left(1+\gamma\right)\left(1-\alpha\right) < 0$, a maximum. Finally, when $\rho = r$, inequality 29 reduces to an equality.

When $\alpha > 1$ the analysis above holds with the following exceptions: the sign of inequality 29 is reversed so that minimisation with respect to ρ is performed; the first and second order conditions are the same, but the latter now corresponds to a minimum at $\rho = r$ as $\alpha > 1$.

Finally, when $\alpha = 1$ (while $T = \tilde{T} = \infty$), welfare expressions 20 and 21 become, by l'Hôpital's rule,

$$\lim_{\alpha \to 1} W_i(k(0)) = \frac{\ln\theta - 1 - (1+\gamma)\frac{\rho - r}{\rho} - \gamma \ln\left[\frac{1}{\rho}\frac{(1+\gamma)r-\rho}{\gamma}\right]}{(1+\gamma)\rho};$$
$$\lim_{\alpha \to 1} \tilde{W}_i(k(0)) = \frac{\ln\theta - 1}{(1+\gamma)\rho}. \tag{30}$$

In this case, $W_i(k(0)) \geq \tilde{W}_i(k(0))$ is equivalent to

$$(1+\gamma)\frac{\rho - r}{\rho} + \gamma \ln\left[\frac{(1+\gamma)r-\rho}{\gamma\rho}\right] \leq 0.$$

Differentiating the left hand side with respect to $\rho < (1+\gamma)r$ reveals a maximum at $\rho = r$, causing the inequality to hold with equality.

2. the condition on $k(0)$ ensures finite extinction under autarky. That on α ensures instantaneous payoffs of $-\infty$ beyond $\tilde{T}$. As consumption is always positive under market access, this is avoided, and the result follows.

3. by Theorem 3, the condition on γ ensures that $T \to 0$ so that

$$W_i(k(0)) \to \frac{1}{1-\alpha}\left[\frac{\alpha^2}{\rho - (1-\alpha)r}\theta^{\frac{1-\alpha}{\alpha}} - \frac{1}{\rho}\right]. \tag{31}$$

As expression 21 reduces to $\tilde{W}_i(k(0)) = \frac{1}{(1-\alpha)\rho}\left[\alpha\theta^{\frac{1-\alpha}{\alpha}} - 1\right]$, the relevant inequality becomes

$$\frac{1}{1-\alpha}\left[\frac{\alpha^2}{\rho - (1-\alpha)r}\theta^{\frac{1-\alpha}{\alpha}} - \frac{1}{\rho}\right] \geq \frac{1}{(1-\alpha)\rho}\left[\alpha\theta^{\frac{1-\alpha}{\alpha}} - 1\right].$$

When $\alpha \neq 1$ this reduces to $r \geq \rho$.

When $\alpha = 1, \gamma = 0$ may be substituted into expression 30 to describe autarkic welfare. That under market access is found by applying L'Hôpital's Rule to expression 31 for

$$\frac{1}{\rho}\left[\ln\theta - \frac{2\rho - r}{\rho}\right].$$

The results follows.

4. $\tilde{W}_i\left(k\left(0\right)\right) \geq W_i\left(k\left(0\right)\right)$ is now equivalent to

$$1 \geq \frac{\left(1-\alpha\right)\left(\rho-r\right)}{\rho-\left(1-\alpha\right)r} \geq \left[\frac{\theta^{\frac{1}{\alpha}}-ak\left(0\right)}{\theta^{\frac{1}{\alpha}}}\right]^{\frac{\rho}{a}}.$$

The first inequality is direct. As the RHS term is positive, a necessary condition for the inequality to hold is that $\rho > r$. The LHS is insensitive to $k\left(0\right)$ while the RHS monotonically declines in it. When $k\left(0\right)=0$, the inequality fails while, when $k\left(0\right)=\frac{1}{a}\theta^{\frac{1}{\alpha}}$, it holds.

5. The condition on $k\left(0\right)$ ensures that T and $\tilde{T}$ are finite. At $\alpha \to 0, \Psi\left(t\right)$ increases monotonically, ensuring a unique extinction date satisfying $k\left(0\right)=\theta^{\frac{1}{\gamma}}\frac{1-e^{-\left(a-g\right)T}}{a-g}$. Combined with expression 20 at $\alpha \to 0$, welfare under capital markets is

$$W_i\left(k\left(0\right)\right)=\frac{\gamma}{1+\gamma}\theta^{\frac{1}{\gamma}}\frac{\gamma}{\left(1+\gamma\right)r-\rho}\left\{1-\left[\frac{\theta^{\frac{1}{\gamma}}-\left(a-g\right)k\left(0\right)}{\theta^{\frac{1}{\gamma}}}\right]^{\frac{r-g}{a-g}}\right\}-\frac{1}{\rho}.$$

Similarly, expression 21 may be rewritten as

$$\tilde{W}_i\left(k\left(0\right)\right)=\frac{1}{\rho}\left\{\left[\frac{\gamma}{1+\gamma}\theta^{\frac{1}{\gamma}}-1\right]\left[1-\left(\frac{\theta^{\frac{1}{\gamma}}-ak\left(0\right)}{\theta^{\frac{1}{\gamma}}}\right)^{\frac{\rho}{a}}\right]-\left(\frac{\theta^{\frac{1}{\gamma}}-ak\left(0\right)}{\theta^{\frac{1}{\gamma}}}\right)^{\frac{\rho}{a}}\right\}.$$

Thus, $\tilde{W}_i\left(k\left(0\right)\right) \geq W_i\left(k\left(0\right)\right)$ is equivalent to

$$\Delta\left(\rho\right) \equiv \left[1-e^{-\rho T}\right]\frac{\gamma\rho}{\left(1+\gamma\right)r-\rho}-\left[1-e^{-\rho\tilde{T}}\right] \leq 0;$$

so that $\Delta\left(r\right)=0$. Differentiation also reveals

$$\Delta'\left(\rho\right)=\left(T+\rho\frac{\partial T}{\partial\rho}\right)e^{-\rho T}\frac{\gamma\rho}{\left(1+\gamma\right)r-\rho}+\left(1-e^{-\rho T}\right)\frac{\left(1+\gamma\right)\gamma r}{\left[\left(1+\gamma\right)r-\rho\right]^2}-\tilde{T}e^{-\rho\tilde{T}};$$

as $\frac{\partial\tilde{T}}{\partial\rho}=0$. Thus

$$\Delta'\left(r\right)=\left[\frac{\theta^{\frac{1}{\gamma}}-ak\left(0\right)}{\theta^{\frac{1}{\gamma}}}\right]^{\frac{r}{a}}\left\{\frac{r}{a^2}\ln\left[\frac{\theta^{\frac{1}{\gamma}}}{\theta^{\frac{1}{\gamma}}-ak\left(0\right)}\right]-\frac{r}{a}\frac{k\left(0\right)}{\theta^{\frac{1}{\gamma}}-ak\left(0\right)}-\frac{1+\gamma}{r}\right\}+\frac{1+\gamma}{r}.$$

When this term is negative, autarky dominates market access for ρ just greater than r. When $a > r, \lim_{k(0)\to\frac{1}{a}\theta^{\frac{1}{\gamma}}}\Delta'\left(r\right)=-\infty$.

The limit in $k\left(0\right)$ is only appropriate if $\frac{1}{a}\theta^{\frac{1}{\gamma}} \leq \mu\phi^*$ when $a > r$ at $\alpha \to 0$. As $\lim_{\alpha\to 0}\phi^*=\frac{1}{a-g}$ when $a > r$ and $\lim_{\alpha\to 0}\mu=\theta^{\frac{1}{\gamma}}$, it is.

6. we first demonstrate equality at $k(0) = 0$. By equation 20, and Lemmas 1 and 3,
$$\lim_{k(0)\to 0} W_i(k(0)) = -\frac{1}{(1-\alpha)\rho};$$
when $\alpha < 1$. Similarly, $\lim_{k(0)\to 0} \tilde{W}_i(k(0)) = -\frac{1}{(1-\alpha)\rho}$.

To show that market access dominates autarky for small $k(0)$ note that $W_i'(k(0)) \geq \tilde{W}_i'(k(0))$ is equivalent to
$$\frac{\gamma}{1+\gamma}\nu^{\frac{1+\gamma}{\alpha+\gamma}}\theta^{\frac{1-\alpha}{\alpha+\gamma}}\frac{e^{-(r-g)T}}{[Q_{r-g}(T)]^{\frac{1+\gamma}{\alpha+\gamma}\alpha}}\frac{dT}{dk(0)} \geq \frac{1}{1-\alpha}\left[1-\frac{\alpha+\gamma}{1+\gamma}\theta^{\frac{1-\alpha}{\alpha+\gamma}}\right]e^{-\rho\tilde{T}}\frac{d\tilde{T}}{dk(0)}.$$

The inequality holds as all terms remain finite and positive as $k(0) \to 0$ except $Q_{r-g}(T)$, which converges to 0.

□

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